The Divine Spark Complexity Law: Actualizing Relations, Fixed-Core Subdirect Dimension, and the Polynomial-to-Exponential Threshold in Mathematical Object-Worlds

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This paper develops the next stage of Theory-Preserving Ontogenesis by formalizing the Divine spark as a minimal system of actualizing relations rather than as an unexplained addition of mathematical law. Building on the July 2026 paper From Divine Logos to Theory-Preserving Ontogenesis: Computational Universal Algebra and the Lawful Emergence of Mathematical Object-Worlds, DOI: 10.13140/RG.2.2.25983.98723, and the accompanying computational repository at https://github.com/DougYouvan/theory-preserving-ontogenesis-v0.8.0-relation-view-degree-, it studies finite semilattice coordinate algebras generated by g primitive elements and presented by s equations. The central theorem bounds their minimum fixed-core subdirect dimension by the first s + 1 binomial levels, while explicit constructions attain polynomial growth of exact degree s + 1. Bounded actualization therefore yields polynomial reconstruction complexity, sublinear actualization yields subexponential complexity, and exponential lawful-view complexity requires defining relations proportional to generator rank. The one-relation case is solved sharply. These results give the Divine spark a quantitative interpretation: a small initiating constraint can reorganize an immense possibility space, but its reconstructive consequences obey an exact complexity law.

Keywords: Divine spark, Divine Logos, theory-preserving ontogenesis, relation-view degree law, fixed-core subdirect dimension, actualizing relations, computational universal algebra, finite semilattices, subdirect representation, separating homomorphisms, generator rank, equation complexity, presentation complexity, reconstruction complexity, mathematical object-worlds, lawful views, polynomial complexity, exponential threshold, binary entropy, join-irreducible elements, union-closed families, algebraic geometry, mathematical inception, artificial intelligence, AI-assisted mathematics.

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1. From Divine Logos to the Complexity of Actualization

1.1 The Previous Transition from Logos to Theory-Preserving Ontogenesis

The earlier development began from a philosophical question: if mathematics is not merely invented but disclosed through intelligible structure, how can that disclosure be represented in exact formal terms? The concept of Divine Logos supplied the metaphysical orientation. Logos was treated not as one mathematical object among others, but as the ground of intelligibility through which relation, order, identity, operation, and law become possible. Mathematics then appeared as a progressive articulation of structures that are lawful before they are fully named, represented, or understood.

Theory-Preserving Ontogenesis translated part of that philosophical program into finite universal algebra. Instead of beginning with a fully specified mathematical world, the computational framework began with a small core algebra and searched for larger structures that preserved the equational theory of that core while admitting faithful reconstruction from homomorphic views. The central question was no longer simply whether a larger algebra could be built. It was whether a larger object-world could arise without violating the law encoded by its originating core.

This changed the mathematical focus from arbitrary construction to lawful emergence. A theory-preserving lift was not merely a larger multiplication table. It was a structure whose operations remained compatible with a prescribed equational regime. Its elements could be understood through quotient maps into the core, and sufficiently many such maps could jointly reconstruct the larger world. The program therefore joined three concerns that are often separated: the preservation of law, the emergence of objecthood, and the recoverability of a world from its lawful perspectives.

The previous stage established that these concerns do not collapse into ordinary information theory. Two finite worlds of the same size may require different numbers of law-preserving views. A four-element carrier requires only two unrestricted binary coordinates for labeling, yet some four-element theory-preserving lifts require three homomorphic views into their two-element core. Later constructions amplified this discrepancy, producing families in which the minimum number of lawful views becomes linear in the number of objects even though the world is generated by only logarithmically many primitive elements.

The transition from Logos to Theory-Preserving Ontogenesis therefore produced a rigorous distinction between bare distinguishability and lawful intelligibility. An object-world may be easy to label but difficult to reconstruct through maps that preserve its governing operation. This distinction supplied the first computationally exact form of the claim that the internal law of a world constrains how that world can be known.

1.2 Mathematical Inception as Progressive Disclosure

Mathematical inception should not be understood as the production of form from absolute formal nothingness. Every mathematical construction presupposes some admissible distinction, operation, inferential rule, or background language. Even the decision to treat two marks as separate objects already invokes identity and difference. The relevant question is therefore not how mathematics arises from complete nonstructure, but how increasingly articulated structure emerges from a prior field of lawful possibility.

The sequence may be written schematically as

Logos → intelligibility → relation → operation → invariant → object → law → theory.

This sequence does not imply that each term occurs temporally after the preceding term. It is an order of dependence. Objects become mathematically stable only when relations and operations permit invariant identification. Laws become expressible only when the relevant objects and operations have been sufficiently resolved. Theories emerge when laws are organized into systems capable of implication, comparison, compression, and extension.

Theory-Preserving Ontogenesis adds two further stages:

theory → actualization → lawful reconstruction.

A theory defines a space of possible structures. It does not, by itself, select one structure as the realized world under consideration. Actualization occurs when additional relations identify, exclude, or constrain possibilities so that a particular quotient structure is selected. Lawful reconstruction then asks how many admissible perspectives are needed to recover that selected world without abandoning the theory through which it arose.

This development separates three senses of mathematical emergence. Generative emergence concerns how elements are produced from primitive generators. Quotient emergence concerns how relations identify parts of a free possibility space to form a definite structure. Reconstructive emergence concerns how the resulting structure becomes recoverable from a family of lawful observations. These three forms of emergence need not have comparable complexity.

A world may require many generators but few views. It may arise from one additional equation yet require hundreds or thousands of lawful views. Conversely, a world may admit a compact subdirect representation while requiring a large set of primitive generators. Mathematical inception is therefore multidimensional. There is no single scalar quantity measuring how difficult a structure is to bring forth, define, or know.

The present paper focuses on this multidimensionality. It asks how the number of actualizing equations controls the minimum number of law-preserving views required for exact reconstruction. The resulting answer reveals a hierarchy in which the formal complexity of actualization governs the possible growth rate of reconstructive complexity.

1.3 Why a Generator Is Not Yet a Divine Spark

The term generator has a precise algebraic meaning. A subset S of an algebra A generates A when every element of A can be obtained from S by repeated application of the algebra's primitive operations. The minimum cardinality of such a set is the generator rank γ(A). This quantity measures how much primitive material must be supplied before operational closure can produce the entire world.

It is tempting to identify a minimal generating set with a Divine spark. The analogy is initially attractive. A small set of generators may unfold into a much larger structure, just as a small initiating cause may appear to produce a vast ordered consequence. Yet this analogy is incomplete.

Generators do not usually select one world from among many worlds allowed by the same law. They populate a free or already specified algebraic environment. Once the operation table or equational theory is fixed, closure of the generators produces elements according to rules already present. The generators function as initial material, not as the principle that determines which quotient of possibility becomes actual.

For example, the free semilattice on g generators contains every nonempty formal meet that can be formed from those generators, subject only to associativity, commutativity, and idempotence. The generators create access to the free possibility space, but they do not choose one nonfree semilattice from the many quotients of that space. To obtain a particular quotient world, one must impose additional identities. These identities identify terms that were distinct in the free structure and thereby change the ontology of the resulting algebra.

The Divine spark, in the present formalization, is therefore better associated with actualizing relation than with generative material. It is not simply x_1, x_2, …, xg. It is an equation such as

u = v,

or a finite system of such equations, which transforms the free algebra into a selected quotient. The spark determines that certain possibilities are not merely similar but identical within the actual world. It changes which distinctions count as real.

This distinction matters because generator complexity and relation complexity can diverge sharply. Two worlds may have the same generator rank while requiring radically different numbers of defining equations. They may also have the same number of defining equations

while exhibiting radically different reconstruction requirements. A mathematically serious account of the spark must therefore distinguish the supply of primitive elements from the imposition of identity-producing constraints.

1.4 The Difference Between Law, Possibility, Selection, and Actualization

Four levels must be kept conceptually separate.

Law specifies the operations and equations that define the general kind of structure under consideration. In the semilattice case, the primitive law includes associativity, commutativity, and idempotence. These axioms determine what counts as a semilattice but do not identify one particular finite semilattice.

Possibility is represented by the relatively free algebra. Given g generators and a background variety, the relatively free algebra contains all formal elements obtainable under the governing law without additional identifications beyond those forced universally. It is the maximally uncommitted world compatible with the chosen primitive theory and generator set.

Selection introduces additional relations that distinguish one quotient from the many quotients available. An equation does not add a new primitive operation. It restricts how the existing operation may differentiate terms. Selection therefore occurs within law rather than outside it.

Actualization is the formation of the quotient structure in which the selected equations hold as identities. Once the congruence generated by the equations is imposed, previously distinct terms may become the same object. The resulting algebra is no longer merely one possible interpretation considered externally. It is the definite mathematical world selected by the presentation.

These four levels may be summarized as

law + generators → free possibility,

free possibility + relations → actual quotient world.

The distinction is analogous to the difference between a space of solutions and one selected solution, between a dynamical law and an initial condition, or between a language and a particular theory expressed in that language. It is also analogous to biological cases in which chemical law permits many molecular organizations, while one inherited code, boundary condition, or regulatory architecture selects a particular self-maintaining organization.

The mathematics developed here concerns the complexity of this selection. How many relations are required to actualize a world whose lawful reconstruction is difficult? Can one equation produce exponentially many indispensable perspectives? Does a bounded amount of actualizing

information permit unlimited representational amplification? These questions move the Divine spark from metaphor toward extremal algebra.

1.5 From Reconstruction Complexity to Actualization Complexity

The earlier work concentrated on reconstruction complexity. Given a finite algebra A and a fixed core B, the lawful-view number $\nu_B(A)$ is the smallest number m for which A admits a separating subdirect embedding into B^m. Equivalently, it is the smallest number of surjective homomorphisms from A to B whose combined values distinguish every pair of elements of A.

This quantity measures the minimum number of lawful binary or finite-core perspectives required to recover the entire world. It differs from the ordinary information-theoretic lower bound. A set of n objects can be labeled by $\lceil \log|B| \, n \rceil$ unrestricted B-valued coordinates, but homomorphic coordinates are constrained by the algebraic operation. A larger number may therefore be necessary.

Reconstruction complexity alone, however, does not explain why a particular world has become difficult to reconstruct. A large value of $\nu_B(A)$ might result from a highly elaborate defining presentation, from a complicated primitive law, or from a small relation that reorganizes the space of admissible views. To identify the source of the complexity, the presentation of the world must be included.

This motivates the actualization number $\sigma_B(A)$. Informally, $\sigma_B(A)$ is the minimum number of additional equations required, at minimum generator rank, to present A as a residual quotient of the relatively free algebra associated with B. It measures how many identity-producing constraints are needed to select the world.

The combined profile is

$(\gamma(A), \sigma_B(A), \nu_B(A))$.

The first coordinate measures primitive generative input. The second measures actualizing relational input. The third measures reconstructive output. The central problem is to determine the possible inequalities among them.

The relation–view degree law establishes that, for semilattice coordinate worlds, these quantities are not independent. If a g-generated world is actualized by s equations, then its lawful-view number cannot exceed the sum of the first s + 1 binomial levels. In particular, fixed s permits at most polynomial growth of degree s + 1 in g. Exponential reconstruction complexity requires s to grow proportionally with g.

Actualization complexity therefore governs the possible asymptotic regime of reconstruction complexity. This is the main mathematical advance developed in the present paper.

1.6 The Computational Deposit and Reproducible Research Program

The results are accompanied by the public computational deposit Theory-Preserving Ontogenesis v0.8.0: Relation–View Degree. The repository contains exact finite constructions, theorem-oriented documentation, independent audit paths, machine-readable summaries, frozen output tables, tests, manifests, and reproduction commands.

The computational role is not to replace proof. It serves four complementary functions.

First, exact enumeration discovers patterns that are not obvious from the definitions. The early four-element atlas revealed the first mismatch between generator rank and lawful-view number. Later searches isolated the three two-state core types capable of reconstruction overhead. These finite patterns guided the formulation of more general theorems.

Second, computation tests conjectures before proof effort is committed. The rank-two and rank-three searches ruled out low-rank counterexamples within complete reduced spaces. The rank-four census exposed extremal semilattice systems that suggested infinite constructions.

Third, independent implementations test the possibility of hidden coding assumptions. The deposit includes multiple routes for checking coordinate closures, join-irreducible counts, homomorphism essentiality, and one-relation formulas. Agreement among structurally different implementations strengthens confidence in the computational claims.

Fourth, the deposit preserves a reproducible record of the transition from experimental mathematics to closed-form theorem. Frozen artifacts permit later researchers to compare regenerated outputs against exact prior results, while slow exhaustive modes remain separated from routine verification.

The repository is not itself the proof of the general theorem. The proof rests on the structure of union-closed solution families, the bounded size of join-irreducible solutions, and explicit extremal constructions. The code supplies verification, examples, boundary testing, and transparency.

1.7 The Central Thesis of the Present Paper

The central thesis is that mathematical actualization obeys a degree law.

For finite semilattice coordinate worlds generated by g primitive elements and actualized by s equations, the minimum number of lawful two-state views satisfies

$v_2(A) \leq \sum$ from j = 1 to s + 1 of C(g,j).

This upper bound is asymptotically sharp for every fixed s. Explicit block-chain presentations with s equations produce lawful-view complexity of order g^(s+1). The number of actualizing

relations therefore determines the highest possible polynomial degree of reconstruction complexity.

Several consequences follow.

With no additional relation, lawful-view complexity is at most linear in generator rank.

With one relation, it may become quadratic but cannot become cubic.

With two relations, cubic growth is possible.

With s fixed relations, degree s + 1 is both the upper limit and an attainable order of growth.

If the number of relations grows sublinearly in g, the lawful-view number remains subexponential.

If exponential lawful-view complexity occurs, the number of actualizing relations must be linear in generator rank.

This theorem gives the Divine spark a disciplined mathematical meaning. A small actualizing cause can produce a consequence vastly larger than itself, but the amplification is not arbitrary. The formal complexity of the initiating constraint determines the asymptotic complexity of lawful reconstruction.

The resulting picture is neither reductionist nor magical. The spark does not contain an explicit list of all objects and perspectives that follow from it. Its consequences unfold through closure under a prior law. Yet the spark is not an unrestricted symbol for mystery. Its relational content can be counted, bounded, and compared with the complexity of the world it selects.

The formal sequence developed in this paper is therefore

Logos → law → free possibility → actualizing relation → quotient world → lawful views → reconstruction.

The Divine spark is located at the transition from possibility to actuality. The relation–view degree law measures how strongly that transition can amplify the complexity of a mathematical world.

2. Standard Universal-Algebraic Formulation

2.1 Finite Algebras, Terms, and Equational Presentation

An algebra consists of a carrier set together with one or more operations of specified arities. In the present development, the principal model is a finite algebra with one binary operation,

although the definitions extend naturally to arbitrary finitary signatures. A binary algebra may be written

$$A = (|\ A\ |, \star),$$

where $|\ A\ |$ is the underlying finite set and

$$\star : |\ A\ | \times |\ A\ | \to |\ A\ |$$

is the primitive operation.

A term is a formal expression constructed from variables using the primitive operations. For a binary signature, examples include

$$x, x \star y, (x \star y) \star z, x \star (y \star x).$$

Every term $t(x_1, \ldots, x_g)$ determines a term operation on A,

$$t^A : A^g \to A,$$

obtained by interpreting the formal operation symbol as the actual operation of A.

An equation is a formal identity

$$u(x_1, \ldots, x_g) = v(x_1, \ldots, x_g)$$

between two terms. The algebra A satisfies this equation when

$$u^A(a_1, \ldots, a_g) = v^A(a_1, \ldots, a_g)$$

for every assignment $(a_1, \ldots, a_g) \in A^g$.

A collection of equations determines an equational class, or variety, consisting of all algebras that satisfy those equations. The semilattice variety is defined by associativity, commutativity, and idempotence:

$$(x \wedge y) \wedge z = x \wedge (y \wedge z),$$
$$x \wedge y = y \wedge x,$$
$$x \wedge x = x.$$

These identities specify the general law of semilattice combination. They do not determine the number of elements, the arrangement of the order, or the particular identifications among formal terms in any one semilattice.

A finite algebra may also be presented by generators and relations. One begins with variables

$$x_1, \dots, x_g$$

and imposes a finite family of equations

$$u_1 = v_1, \dots, u_s = v_s.$$

The resulting presentation is written schematically as

$$A = \langle x_1, \dots, x_g \mid u_1 = v_1, \dots, u_s = v_s \rangle.$$

The generators specify primitive points from which all elements of Aarise by term formation. The relations specify which formal terms are to be identified. The presentation therefore separates two distinct contributions to the world:

generative supply and identity-producing constraint.

This separation is fundamental to the present theory. Generator rank measures how many primitive elements are required. Relation count measures how many additional equations are needed to select the quotient world. Neither quantity alone determines the minimum number of lawful views required to reconstruct the resulting algebra.

A presentation need not be minimal. The same algebra may admit many generating sets and many systems of defining equations. Some equations may be consequences of others. Some generators may be expressible in terms of the remaining generators. The complexity quantities developed later are therefore defined by minimization over admissible presentations rather than by counting the symbols appearing in one arbitrary description.

2.2 Relatively Free Algebras

Let $\mathcal{V}$be a variety of algebras and let

$$X_g = \{x_1, \dots, x_g\}$$

be a set of $g$formal generators. The relatively free algebra

$$F_{\mathcal{V}}(g)$$

is the algebra of terms in those generators, modulo all identities valid throughout $\mathcal{V}$.

The phrase relatively free emphasizes that the algebra is free only relative to the governing equational theory. Terms that are universally equal in the variety have already been identified. Terms not forced to be equal remain distinct.

The defining universal property is as follows. Given any algebra $A \in \mathcal{V}$and any assignment

$$f: X_g \to A,$$

there exists a unique homomorphism

$$\hat{f}: F_{\mathcal{V}}(g) \to A$$

extending the assignment. Thus every g-generated algebra in the variety is a homomorphic image of $F_{\mathcal{V}}(g)$.

The relatively free algebra represents maximal possibility under fixed law and a fixed number of generators. It contains every term-generated distinction that is not universally forbidden by the theory. A particular finitely generated algebra is obtained by identifying some of these distinctions through a congruence.

For the semilattice variety, every term is equivalent to the meet of the distinct variables that appear in it. Repetition and order are irrelevant because of idempotence and commutativity, and parenthesization is irrelevant because of associativity. Consequently, the free semilattice on $g$generators may be represented by the nonempty subsets of $[g]$:

$$[g] = \{1, \dots, g\}.$$

The subset $S \subseteq [g]$, $S \neq \emptyset$, represents the term

$$\bigwedge_{i \in S} x_i.$$

The semilattice operation is union:

$$S \wedge T = S \cup T,$$

when the order is taken to be reverse inclusion. The free semilattice therefore has

$$| F_{\mathrm{SL}}(g) | = 2^g - 1$$

elements.

This representation is especially important because it turns algebraic equations into combinatorial constraints on subsets. A semilattice equation

$$\bigwedge_{i \in P} x_i = \bigwedge_{j \in Q} x_j$$

identifies the free elements represented by $P$and Q. Its consequences identify further subsets after closure under union. The actual quotient world is thus obtained from a Boolean-like possibility space by imposing a congruence generated by selected identifications.

The relatively free algebra is not merely a convenient technical object. It provides the formal distinction between law and actualization. The laws of the variety determine $F_{\mathcal{V}}(g)$. Additional relations select a quotient of it.

2.3 Congruences Generated by Equations

A congruence on an algebra $F$is an equivalence relation compatible with every primitive operation. For a binary algebra, an equivalence relation $\theta$is a congruence when

$$a \equiv_\theta a', b \equiv_\theta b'$$

implies

$$a \star b \equiv_\theta a' \star b'.$$

Compatibility ensures that operations remain well defined after equivalent elements are identified.

Given a set of equations

$$R = \{u_1 = v_1, \dots, u_s = v_s\}$$

in the relatively free algebra, there exists a least congruence containing every pair

$$(u_i, v_i).$$

It is denoted

$$\mathrm{Cg}(R).$$

The quotient algebra

$$F_V(g)/\mathrm{Cg}(R)$$

is the algebra presented by the generators and relations.

The adjective least is essential. Actualization does not introduce arbitrary identifications beyond those forced by compatibility and transitivity. Starting from the explicitly imposed equations, one closes under reflexivity, symmetry, transitivity, substitution into operations, and all consequences of the background theory. The quotient contains exactly the identifications required by the chosen relations and no more.

This closure process explains why a small relation set can have a large structural effect. One equation may propagate through many contexts. If

$$u = v,$$

then for every term context $C[-]$,

$$C[u] = C[v]$$

must also hold. In a semilattice, joining both sides with another term produces further identifications:

$$u \wedge w = v \wedge w.$$

The spark is therefore not a list of all consequences. It is a compact source from which consequences propagate through lawful closure.

Different relation sets may generate the same congruence. If

$$\mathrm{Cg}(R) = \mathrm{Cg}(R'),$$

then they define isomorphic quotients of the same relatively free algebra. The significant object is ultimately the generated congruence, not the syntactic form of one presentation.

Nevertheless, the minimum size of a relation set generating the relevant congruence contains useful information. It measures the smallest number of explicit identifications required to actualize the quotient at the chosen generator rank. This motivates the actualization number developed in Section 3.

Congruence generation also reveals an important asymmetry. A relation set may be syntactically small while the resulting congruence has many equivalence classes containing many identified terms. The complexity of the cause, measured as a minimal basis, need not resemble the number of its consequences.

2.4 Quotient Worlds as Actualized Mathematical Structures

Let

$$F = F_{\mathcal{V}}(g)$$

be the relatively free algebra and let

$$\theta = \mathrm{Cg}(R)$$

be the congruence generated by a set of additional equations. The quotient

$$A = F/\theta$$

consists of equivalence classes

$$[t]_\theta$$

of terms.

The quotient operation is defined by

$$[t]_\theta \star [s]_\theta = [t \star s]_\theta.$$

Congruence compatibility guarantees that this definition does not depend on the representatives chosen.

The quotient is an actualized mathematical structure in the limited formal sense used here. Before the relations are imposed, the relatively free algebra retains every distinction allowed by the background theory. After the relations are imposed, certain distinctions cease to exist within the quotient. Terms that were formerly separate become one object.

This is not temporal creation. The quotient algebra is mathematically definable whenever its presentation is definable. Actualization refers instead to the formal selection of a particular identity structure from a space of possible identity structures.

The lattice of congruences of Forganizes these possibilities. The least congruence produces the free algebra itself. The greatest congruence collapses all elements into one. Intermediate congruences produce intermediate quotient worlds.

If

$$\theta \subseteq \phi,$$

then F/ϕ contains at least as many identifications as F/θ. Thus the congruence lattice is an ordering of possible resolutions. Finer congruences preserve more distinctions. Coarser congruences actualize worlds with fewer distinct objects.

However, carrier size alone does not determine structural complexity. Two quotients with the same number of elements may differ in their homomorphism sets, automorphism groups, subalgebra lattices, and minimum subdirect dimensions. Actualization changes not only the number of objects but the geometry of lawful observation.

This observation is central to the one-relation discontinuity theorem. A single equation can alter only a vanishing fraction of the carrier relative to the free algebra, yet dramatically increase the number of fixed-core views required for reconstruction. The quotient may remain nearly as large as the free world while becoming much harder to recover through homomorphic observation.

The decisive change lies not in the number of surviving objects but in the pattern of admissible identifications among them.

2.5 Subdirect Embeddings into a Fixed Core

Let $A$and Bbe algebras of the same signature. An embedding

$$e: A \hookrightarrow B^m$$

is subdirect when every coordinate projection

$$\pi_i \circ e: A \to B$$

is surjective.

Equivalently, a subdirect embedding is determined by a family of surjective homomorphisms

$$h_1, \dots, h_m : A \to B$$

such that the diagonal map

$$\Delta_H : A \to B^m, \Delta_H(a) = (h_1(a), \dots, h_m(a))$$

is injective.

Injectivity means that distinct elements of $A$are separated by at least one coordinate:

$$a \neq b \Longrightarrow \exists i \; h_i(a) \neq h_i(b).$$

Surjectivity means that each coordinate genuinely realizes the full core B, rather than landing in a proper subalgebra.

The algebra $A$is then represented as a subalgebra of a direct power of B. Every operation in $A$is performed coordinatewise in B^m. The representation is therefore law preserving.

In classical universal algebra, subdirect decompositions represent algebras through subdirectly irreducible factors or through members of a specified generating class. In the present work, the factor algebra $B$is fixed in advance. The question is not merely whether Ahas some subdirect decomposition, but how many copies of the prescribed core are necessary.

This fixed-core condition is essential. A larger or more expressive factor could encode the same algebra in fewer coordinates. The lawful-view number measures representation relative to one chosen primitive law.

For a two-element core, every coordinate carries one binary value. An arbitrary set of $n$objects can be injectively labeled by

$$\lceil \log_2 n \rceil$$

binary coordinates. A fixed-core subdirect embedding may require more because each coordinate must be a homomorphism. It cannot assign labels arbitrarily.

The difference

$$\nu_B(A) - \left\lceil \log_{|B|} | A | \right\rceil$$

measures fixed-core representation overhead. It quantifies how much extra perspectival complexity is imposed by the requirement that every view preserve the operation.

2.6 Separating Families of Surjective Homomorphisms

Let

$$\mathrm{SurHom}(A, B)$$

denote the set of all surjective homomorphisms from Aonto B. A subset

$$H \subseteq \mathrm{SurHom}(A, B)$$

is separating when for every distinct $a, b \in A$, some $h \in H$satisfies

$$h(a) \neq h(b).$$

The intersection of the kernels of the maps in $H$is then the equality relation:

$$\bigcap_{h \in H} \ker h = 0_A.$$

Here 0_Adenotes the diagonal congruence on A.

Thus separating families may be described in three equivalent ways:

injective diagonal map,
intersection of kernels equal to identity,
every pair separated by a lawful coordinate.

These formulations emphasize different aspects of the same structure. The diagonal-map form highlights representation. The kernel form highlights congruence refinement. The pair-separation form highlights observation.

A separating family is irredundant when removing any one of its homomorphisms destroys separation. Each coordinate then has at least one private witness pair

$$(a_h, b_h)$$

separated by $h$but not by the remaining maps.

Minimum separating families are automatically irredundant. Their members represent indispensable lawful perspectives.

The existence of a separating family of surjective maps means that $A$is residually B. For every distinct $a, b \in A$, there exists a homomorphism onto Bseparating them. Residuality is therefore the condition under which exact reconstruction from B-valued lawful views is possible at all.

If $A$is not residually B, then some distinct elements remain indistinguishable under every homomorphism into B. No number of B-valued lawful views can reconstruct A. The best possible observation recovers only the quotient by the intersection of all kernels.

The present paper restricts its principal dimension results to residual coordinate algebras, where reconstruction is possible and the minimum number of required views is finite.

2.7 Fixed-Core Subdirect Dimension

The fixed-core subdirect dimension of a finite residual algebra Arelative to $B$is

$$\nu_B(A) = \min\{m : A \hookrightarrow B^m \text{ subdirectly}\}.$$

Equivalently,

$$\nu_B(A) = \min\left\{ |\, H \,| : H \subseteq \mathrm{SurHom}(A, B),\ \bigcap_{h \in H} \ker\, h = 0_A \right\}.$$

The term lawful-view number is used as an interpretive synonym. The standard-language formulation should remain primary: $\nu_B(A)$is the minimum number of copies of the fixed core required in a separating subdirect representation.

The dimension is relative to the core. The same algebra may satisfy

$$\nu_B(A) \neq \nu_C(A)$$

for different factor algebras $B$and C. A core with richer operations may permit more efficient representation. A weak or asymmetric core may require many more coordinates.

The most elementary lower bound comes from cardinality:

$$|\, A \,| \leq |\, B \,|^{\nu_B(A)}.$$

Hence

$$\nu_B(A) \geq \lceil \log_{|B|} | A | \rceil.$$

Equality means that the algebra is representable with no overhead beyond ordinary information-theoretic labeling. Strict inequality means that law-preserving coordinates are less efficient than arbitrary coordinates.

A universal finite upper bound is

$$\nu_B(A) \leq | A | - 1.$$

To see this, fix one reference element $a_0 \in A$. For each remaining element $a \neq a_0$, choose a homomorphism separating $a$from a_0. A suitable refinement argument produces a separating family with at most $| A | - 1$members.

If $A$is generated by $g$elements and $| B | = 2$, another bound is

$$\nu_B(A) \leq 2^g,$$

because a homomorphism is determined by its values on a generating set, and there are at most 2^gbinary value patterns.

The relation–view degree theorem introduces a stronger bound when presentation complexity is known:

$$\nu_2(A) \leq \sum_{j=1}^{s+1} \binom{g}{j}.$$

This transforms relation count into a structural bound on fixed-core subdirect dimension.

2.8 Standard Terminology and the Lawful-View Interpretation

The terminology of lawful views is not intended to conceal the classical structure. A lawful view is a surjective homomorphism onto the fixed core. A complete family of lawful views is a separating family. Reconstruction is an injective diagonal representation. The view number is a fixed-core subdirect factor count.

The interpretive language is useful because it emphasizes a question that standard terminology does not always foreground: how many operation-preserving perspectives are needed to recover a world exactly?

An arbitrary binary coordinate may label elements without respecting any structure. A homomorphism must satisfy

$$h(a \star b) = h(a) \star h(b).$$

Its outputs are constrained by the operation. The coordinate therefore records not only object identity but compatibility with law.

The lawful-view interpretation also clarifies why additional coordinates may be necessary. The issue is not a shortage of raw bits. It is a shortage of admissible bits. Many binary labelings are forbidden because they would assign outputs incompatible with the multiplication table.

This distinction is analogous to measurement under invariance constraints. An arbitrary code can distinguish states by external convention. A lawful observable must interact coherently with the system's transformations. The number of unconstrained labels and the number of admissible observables need not agree.

The philosophical significance should nevertheless be stated carefully. A homomorphism is a mathematical map, not necessarily a physical observer or conscious perspective. The word view expresses structural projection, not subjectivity. The results concern exact representation within universal algebra.

The lawful-view interpretation becomes productive when joined to actualization complexity. A small equation basis can select a quotient whose admissible homomorphisms have a highly fragmented separation geometry. The number of perspectives then grows not because the world has many more objects, but because lawful maps have become individually less capable of separating them.

2.9 Relation to Classical Universal Algebra

The formal setting rests on established universal-algebraic foundations: varieties, free algebras, congruence generation, quotient presentations, homomorphisms, direct products, subdirect embeddings, residuality, and finite representation.

The present contribution does not claim to invent these concepts. Its aim is to organize them around a quantitative comparison that is less standard:

generator complexity,relation complexity,fixed-core representation complexity.

Classical subdirect representation asks how an algebra decomposes into simpler factors. The present problem fixes the factor $B$and minimizes the number of copies required.

Classical finite presentation theory asks how economically an algebra can be specified by generators and relations. The present problem compares that specification cost with the minimum size of a separating family of B-valued homomorphisms.

Classical universal algebraic geometry studies equations over an algebra, their solution sets, radical congruences, and coordinate algebras. The present framework interprets a minimal equation basis as actualizing constraint and a minimal separating point family as reconstructive requirement.

Classical semilattice theory identifies representations through prime filters, ideals, or two-element factors. The present results ask how the number of defining equations limits the number of indispensable factors.

The novelty claim must therefore be located in the relation among familiar objects rather than in the objects separately. The principal theorem asserts that, in the two-element semilattice setting, $s$defining equations limit indispensable fixed-core factors to the first $s + 1$binomial levels, and that this polynomial degree is sharp.

This produces a quantitative law of presentation-to-representation amplification:

$$s \longmapsto \text{maximum degree } s + 1.$$

The result converts the metaphor of a small actualizing spark into a precise extremal problem. It asks not whether a quotient can have complicated consequences, but how complicated those consequences can be as a function of the size of a minimal relation basis.

That question provides the bridge from standard universal algebra to a general mathematics of actualization.

3. The Spark–View Complexity Triple

3.1 Generator Rank γ(A)

Let Abe a finite algebra. A subset $S \subseteq A$generates $A$when the smallest subalgebra containing $S$is all of A. Equivalently, every element of Acan be written as the value of some term whose variables are assigned elements of S. The generator rank is

$$\gamma(A) = \min\{|\ S\ |: \langle S \rangle_A = A\}.$$

The notation $\langle S \rangle_A$ denotes closure of S under the primitive operations of A.

Generator rank measures the minimum primitive supply from which the algebra can unfold by lawful operational closure. It is a property of the realized algebra, not merely of one chosen presentation. A displayed generating set may contain redundant elements, but $\gamma(A)$ records the smallest possible number.

For a semilattice, generation has a direct combinatorial interpretation. If

$$S = \{a_1, \dots, a_g\},$$

then every generated element is a finite meet of members of S. Idempotence ensures that repeated use of one generator adds nothing, commutativity removes ordering, and associativity removes parenthesization. At most

$$2^g - 1$$

distinct nonempty meets can therefore arise. Consequently,

$$| A | \leq 2^{\gamma(A)} - 1$$

for every finite semilattice generated without a distinguished nullary constant. Hence

$$\gamma(A) \geq \lceil \log_2(| A | + 1) \rceil.$$

The free semilattice on g generators attains the upper carrier bound exactly. Its elements correspond to the nonempty subsets of the generator set.

Generator rank alone does not measure the complexity of the multiplication table, the number of defining relations, or the difficulty of reconstruction. A four-element projection algebra may require all four elements as generators because its operation creates no new carrier elements. A four-element functionally rich algebra may be generated by one element. Yet both may admit reconstruction from only two lawful binary coordinates.

This divergence appeared in the finite atlas preceding the present work. Among the sixteen classified four-element residual lifts, generator ranks ranged from one to four, while lawful-view numbers were two or three. The resulting spark–view profiles showed that operational generativity and homomorphic reconstructibility are independent dimensions.

The word spark was initially associated with generator rank because a small seed can generate a large closure. That interpretation remains useful at one level. A generator is a local beginning

from which the operation develops further elements. However, it does not yet capture actualization. The same generators in the same free algebra permit many different quotient worlds. Generator rank measures how much primitive material is needed after the identity structure of the world has been fixed; it does not measure what selects that identity structure.

Thus $\gamma(A)$is the first coordinate of the complexity triple, but it is not the whole formal meaning of the Divine spark.

3.2 Actualization Number σB(A)

Fix a finite core algebra B, and let

$$\mathcal{V}(B)$$

denote the variety generated by B. Let

$$F_B(g)$$

be the relatively free algebra on $g$generators in that variety. A finite g-generated algebra $A \in \mathcal{V}(B)$is represented by a quotient

$$A \cong F_B(g)/\theta$$

for some congruence θon $F_B(g)$.

Suppose a set of equations

$$R = \{u_1 = v_1, \dots, u_s = v_s\}$$

generates θ. Then Rprovides an equational presentation of Aat generator rank g.

The actualization number relative to $B$is defined schematically by

$$\sigma_B(A) = \min\{|\ R\ |: A \cong F_B(\gamma(A))/\mathrm{Cg}(R)\},$$

subject to the residual and presentation conditions appropriate to the fixed-core setting.

The restriction to $F_B(\gamma(A))$matters. Without it, one could sometimes trade additional generators for fewer relations or fewer generators for more elaborate relations. The present invariant asks for the minimum number of additional equations once primitive generative input has already been minimized.

The value

$$\sigma_B(A) = 0$$

means that $A$is relatively free at its minimum generator rank. No further actualizing equation is required.

The value

$$\sigma_B(A) = 1$$

means that one additional identity suffices to select $A$from the free possibility space.

More generally, $\sigma_B(A) = s$means that $s$equations suffice and that no smaller equation basis presents the same algebra at minimum generator rank.

Actualization number is not the total number of equations true in A. Every finite algebra satisfies infinitely many syntactic equations. Nor is it the number of all pairs identified by the quotient congruence. It is the minimum basis size from which the full congruence follows by equational closure.

This distinction is essential. One equation may generate a congruence containing a vast number of identified term pairs. The explicit cause is small, while the deductive closure is large. The actualization number measures the compact source of the identification pattern rather than its full extension.

The notion is relative to the background core because the relatively free algebra and its universal identities depend on B. An equation that is informative over one core may be redundant over another. Similarly, two syntactically different equations may induce the same congruence in one variety and different congruences in another.

For the two-element semilattice core, the free algebra has a particularly transparent form, and additional equations become identifications among subsets of the generator set. This makes $\sigma_2(A)$amenable to exact extremal analysis.

Actualization number formalizes the transition from lawful possibility to one selected quotient world. It is the second coordinate of the complexity triple and the quantity most closely associated here with the Divine spark.

3.3 Fixed-Core View Number vB(A)

Let Abe finite and residually B. The fixed-core view number is

$$\nu_B(A) = \min\{m : A \hookrightarrow B^m \text{ subdirectly}\}.$$

Equivalently,

$$\nu_B(A) = \min\left\{ |\, H \,| : H \subseteq \mathrm{SurHom}(A,B), \bigcap_{h \in H} \ker\, h = 0_A \right\}.$$

Each $h : A \to B$ is a lawful view because it preserves every primitive operation. The family H reconstructs A when its combined outputs distinguish every pair of elements.

The term fixed-core subdirect dimension is the standard mathematical description of the invariant. Lawful-view number is the interpretive description.

The minimum is constrained below by carrier information:

$$\nu_B(A) \geq \lceil \log_{|B|} |\, A \,| \rceil.$$

If B has two elements, this becomes

$$\nu_2(A) \geq \lceil \log_2 |\, A \,| \rceil.$$

Equality means that operation-preserving coordinates are as efficient as arbitrary labels. Strict inequality means that lawful reconstruction has overhead.

That overhead can be extreme. The semilattice families developed in the preceding release include g-generated worlds whose carrier size is exponential in g, whose ordinary binary dimension is therefore of order g, but whose lawful-view number is proportional to the entire carrier size. In such cases,

$$\gamma(A) = \Theta(\log |\, A \,|), \nu_2(A) = \Theta(|\, A \,|).$$

The present paper adds a missing variable. It asks how many actualizing equations are needed before such view complexity can arise.

The answer is sharply stratified. If $\sigma_2(A) = s$, then in the semilattice setting,

$$\nu_2(A) \leq \sum_{j=1}^{s+1} \binom{\gamma(A)}{j}.$$

Thus view number is not controlled by carrier size or generator rank alone. It is also constrained by actualization complexity.

The view number is the third coordinate of the complexity triple. It measures the minimum lawful observational complexity of the selected world.

3.4 Primitive Input, Defining Constraint, and Reconstruction

The three coordinates have distinct meanings:

$\gamma(A) =$ minimum primitive generative input,
$\sigma_B(A) =$ minimum actualizing relational input,
$\nu_B(A) =$ minimum lawful reconstructive output.

Together they form the Spark–View complexity triple

$$\boxed{\mathcal{C}_B(A) = (\gamma(A), \sigma_B(A), \nu_B(A)).}$$

The first quantity asks:

How many primitive elements are needed for operational closure to generate the world?

The second asks:

How many additional identities are needed to select that world from the corresponding free possibility space?

The third asks:

How many fixed-core homomorphic perspectives are needed to recover the selected world exactly?

These questions cannot be reduced to one another.

A small generator rank does not imply a small actualization number. A world may be generated by two elements but require many independent relations to specify its quotient structure.

A small actualization number does not imply a small lawful-view number. The one-relation theorem produces worlds with

$$\sigma_2(A) = 1$$

and

$$\nu_2(A) = \Theta(\gamma(A)^2).$$

A large generator rank does not imply a large view number. Projection algebras can require every carrier element as a generator yet still admit information-theoretically optimal coordinate representations.

A large carrier does not determine any one of the three quantities. Carrier size supplies lower or upper bounds in some signatures, but structures of equal size can have widely different complexity triples.

The triple therefore records three stages of formal becoming:

$$\text{seed} \rightarrow \text{selection} \rightarrow \text{recovery}.$$

The first stage is constructive. The second is quotient-forming. The third is representational.

The Divine spark interpretation belongs primarily to the middle stage. The spark is the identity-producing constraint that selects one ontology from lawful possibility. Yet its significance is revealed through the relationship between all three coordinates. The central phenomenon is not merely that a small relation exists, but that a small relation can reorganize the requirements of lawful reconstruction.

3.5 Isomorphism Invariance of the Complexity Triple

A meaningful structural invariant must not depend on arbitrary labels.

Suppose

$$A \cong A'.$$

An isomorphism carries generating sets of $A$to generating sets of $A'$and preserves generated subalgebras. Therefore,

$$\gamma(A) = \gamma(A').$$

For fixed B, an isomorphism also transports separating homomorphism families. If

$$\varphi: A \rightarrow A'$$

is an isomorphism and

$$h: A \rightarrow B$$

is a surjective homomorphism, then

$$h \circ \varphi^{-1}: A' \to B$$

is a surjective homomorphism. Separation is preserved, so

$$\nu_B(A) = \nu_B(A').$$

Actualization number requires slightly greater care because it is defined through minimum presentations. Isomorphic algebras may be presented by different congruences on different chosen free algebras. Nevertheless, once generator rank and the background variety are fixed, isomorphic quotients admit corresponding finite presentations. Minimizing over all such presentations yields

$$\sigma_B(A) = \sigma_B(A').$$

Hence

$$\mathcal{C}_B(A) = \mathcal{C}_B(A')$$

for isomorphic residual algebras.

The triple is also invariant under isomorphism of the core-lift pair. If

$$B \cong B',$$

then corresponding homomorphisms into $B$and B'give the same minimum view number after transport through the core isomorphism. The relevant object is therefore the isomorphism type of the pair (A, B), not a particular truth-table labeling.

This point was important in the classification of the sixteen labeled binary operations on a two-element set. Complementation of the carrier pairs some operations into common isomorphism types. The resulting ten core types, not the sixteen raw truth tables, form the natural classification level.

Isomorphism invariance allows the complexity triple to enter a structural taxonomy. Algebras can be grouped by common values of

$$(\gamma, \sigma, \nu),$$

and those profiles can be compared across core types, signatures, and varieties.

However, the triple is not a complete invariant. Distinct nonisomorphic algebras can share the same three values. It is a complexity profile rather than a full classification. Additional data may include automorphism groups, congruence lattices, spark hypergraphs, view hypergraphs, term clones, or orbit structures.

The role of the triple is to isolate the three quantities central to actualization and reconstruction.

3.6 Free Possibility and Actual Quotient Worlds

For a fixed core $B$and generator number g, the relatively free algebra

$$F_B(g)$$

represents the broadest world permitted by the universal identities of B. Every g-generated algebra in the variety generated by $B$is a quotient of this free object.

The free algebra is possibility in a precise categorical sense. Any assignment of the free generators into a target algebra extends uniquely to a homomorphism. It contains no additional identifications beyond those required by the common theory.

The complexity triple of the free algebra has the form

$$\mathcal{C}_B(F_B(g)) = (g, 0, \nu_B(F_B(g))).$$

In the two-element semilattice case,

$$\nu_2(F_2(g)) = g.$$

The $g$coordinate evaluations corresponding to the generators separate all nonempty subsets. Thus

$$\mathcal{C}_2(F_2(g)) = (g, 0, g).$$

An actual quotient world is obtained by choosing a congruence $\theta$and forming

$$A = F_B(g)/\theta.$$

A naive expectation might be that quotienting simplifies representation. Since quotienting reduces distinctions, perhaps fewer views should always suffice. The one-relation theorem shows that this intuition is false under the fixed-core requirement.

A quotient may have fewer elements than the free algebra yet require many more fixed-core views. The reason is that quotienting changes which homomorphisms remain admissible. Some coordinates that were independently useful in the free algebra become linked by the new identity. Separation must then be distributed across a larger family of individually constrained maps.

Thus actualization may simultaneously reduce the number of objects and increase the difficulty of lawful reconstruction.

This is one of the most important structural discoveries of the program. Complexity does not move monotonically with carrier size under quotient formation. A quotient can be ontologically smaller but perspectivally harder.

In the balanced one-relation family, the quotient differs from the free semilattice on only a vanishing fraction of its carrier as $g$grows. Nevertheless, its view number grows quadratically rather than linearly. The actualizing equation creates a global dependency among admissible views whose effect is much larger than the fraction of elements identified.

The distinction between free possibility and actual quotient world therefore supports a precise discontinuity:

small congruence basis ⇒/small change in representation complexity.

3.7 Galois Duality Between Equations and Points

Universal algebraic geometry provides a natural duality between equation systems and their solution sets.

Let $F_B(g)$be relatively free on $g$generators. For a set of equations R, define the solution set

$$V_B(R) \subseteq B^g$$

to consist of all assignments satisfying every equation in R.

Conversely, for a set of points

$$X \subseteq B^g,$$

define the radical congruence

$$\mathrm{Rad}_B(X)$$

to contain every term pair (u, v)whose evaluations agree at every point of X.

These assignments reverse inclusion:

$$R \subseteq R' \Longrightarrow V_B(R') \subseteq V_B(R),$$

and

$$X \subseteq X' \Longrightarrow \mathrm{Rad}_B(X') \subseteq \mathrm{Rad}_B(X).$$

The coordinate algebra of $X$is

$$\Gamma_B(X) = F_B(g)/\mathrm{Rad}_B(X).$$

When $A$is residually B, it can be represented as such a coordinate algebra. Points in Xcorrespond to homomorphisms from $A$to B. A sufficiently rich point set separates the coordinate functions and therefore reconstructs A.

This produces two complementary bases.

An equation basis selects the world from the side of constraint.

A point basis reconstructs the world from the side of observation.

The actualization number measures the minimum size of the first basis. The view number measures the minimum size of the second.

Schematically,

small relation basis ⟷ possibly large point basis.

The relationship is not a simple equality. The principal theorems of this paper show that a system of $s$equations may require a point basis of order g^{s+1}, while no system of $s$semilattice equations can force asymptotically larger polynomial degree.

The Divine spark problem can therefore be restated in Galois-dual form:

How small can the defining equation basis be while the minimum reconstructing point basis becomes large?

For one equation, the answer is quadratic.

For fixed s, the answer has degree $s + 1$.

For sublinear s, the point basis remains subexponential.

For exponential point-basis complexity, $s$must be linear in g.

The spark–view duality is therefore a quantitative refinement of the classical equations–points correspondence.

3.8 The Actualization–Observation Asymmetry

The actualization–observation asymmetry is the possible divergence between

$$\sigma_B(A)$$

and

$$\nu_B(A).$$

A world may be selected by a very small relation basis while requiring a much larger family of lawful observations for exact recovery.

The asymmetry is visible already in the one-relation case. For balanced parameters,

$$\sigma_2(A) = 1$$

while

$$\nu_2(A) = \Theta(\gamma(A)^2).$$

The relation is not itself a coordinate description of the world. It changes the equivalence structure from which admissible coordinates emerge. Its consequences propagate through the algebraic law, and the resulting homomorphism geometry may be highly distributed.

This distinction resembles the difference between specifying a rule and enumerating all outcomes of the rule. A compact recursion may define a large sequence. A short program may generate a complex image. A concise boundary condition may determine an extended field configuration. Yet the present result is more specific: it relates the number of identity-producing

equations to the minimum number of homomorphic points needed to separate a coordinate algebra.

The asymmetry should not be exaggerated into creation of arbitrary complexity from nothing. The background law, the generator set, and the syntactic content of the equations all contribute. Counting equations alone does not measure equation length. One equation may mention many generators and therefore contain extensive internal structure.

For this reason, the relation–view degree theorem should be interpreted as a bound at fixed generator rank and equation count, not as a complete measure of causal or descriptive complexity.

Nevertheless, the theorem establishes a genuine amplification law. Even when the number of displayed relations is fixed, their deductive closure can force a reconstructing family whose size grows as a higher-degree polynomial in the number of generators.

The asymmetry is therefore disciplined rather than unlimited:

$$\sigma = s \Longrightarrow \nu = O(g^{s+1})$$

for fixed $s$in the semilattice setting.

Each additional independent relation can raise the maximum possible polynomial degree by one.

The Divine spark becomes not a suspension of law but a compact input whose consequences are amplified through law.

3.9 Extremal Functions for Spark and View Complexity

The natural general problem is to maximize lawful-view complexity subject to bounds on generative and actualizing complexity.

For a fixed core B, define

$$M_B(g, s) = \max\{\nu_B(A) : \gamma(A) = g, \sigma_B(A) \leq s, A \text{ finite and residually } B\},$$

whenever the maximum exists.

This extremal function asks:

Among all g-generated worlds actualizable by at most $s$relations, how many fixed-core perspectives can be forced?

For the two-element semilattice core, the relation–view degree theorem gives

$$M_2(g,s) \le \sum_{j=1}^{s+1} \binom{g}{j}.$$

The block-chain constructions provide the lower bound

$$M_2(g,s) = \Omega(g^{s+1})$$

for fixed s. Therefore,

$$M_2(g,s) = \Theta(g^{s+1})$$

as $g \to \infty$with $s$fixed.

For $s = 0$,

$$M_2(g,0) = g.$$

For $s = 1$, the complete extremal theorem gives the exact value

$$M_2(g,1) = \left\lfloor \frac{g^2+g+1}{3} \right\rfloor$$

for the relevant one-equation semilattice coordinate worlds.

For larger fixed s, the exact leading constant and exact finite formula remain open. The general upper bound and block lower bound identify the degree but not necessarily the sharp coefficient.

A second extremal function may include carrier size:

$$M_B(n,g,s) = \max\{\nu_B(A) : |\, A \,| = n, \gamma(A) = g, \sigma_B(A) \le s\}.$$

This version separates representation overhead from raw scale. It asks how difficult lawful reconstruction can become when the number of objects, generators, and relations are all controlled.

A third direction minimizes actualization complexity for a prescribed view requirement:

$$S_B(g, v) = \min\{\sigma_B(A) : \gamma(A) = g, \nu_B(A) \geq v\}.$$

The entropy threshold implies that achieving

$$v = 2^{\Omega(g)}$$

requires

$$S_2(g, v) = \Omega(g)$$

in the semilattice case.

These extremal functions turn the Spark–View triple into a research program. Rather than asking only whether small causes can have large effects, they ask for the exact boundary of possible amplification.

The Divine spark is thereby translated into a quantitative question:

How much lawful world-complexity can a bounded actualizing basis force?

The remaining sections solve this question for finite semilattice coordinate worlds at the level of polynomial degree, solve the one-relation case exactly, and identify the transition at which exponential reconstruction becomes possible.

4. Semilattice Coordinate Worlds

4.1 The Two-Element Semilattice Core

The principal core in the present paper is the two-element semilattice

$$\mathbf{2} = (\{0,1\}, \wedge),$$

with operation table

$$0 \wedge 0 = 0, 0 \wedge 1 = 0, 1 \wedge 0 = 0, 1 \wedge 1 = 1.$$

This is the smallest nontrivial semilattice. Its simplicity makes it an ideal fixed core for studying the relation between presentation complexity and reconstruction complexity.

A homomorphism

$$h: A \to \mathbf{2}$$

from a semilattice Amust satisfy

$$h(a \wedge b) = h(a) \wedge h(b).$$

The inverse image of 1,

$$F_h = h^{-1}(1),$$

is closed under meet. In order-theoretic language, it behaves as a filter under the appropriate semilattice convention. The inverse image of 0forms the complementary region in which at least one of the required conditions for membership in $F_h$fails.

Each homomorphism therefore partitions Ainto two lawful classes:

$$h^{-1}(0)\text{and}h^{-1}(1).$$

Unlike an arbitrary binary labeling, this partition must respect the semilattice operation.

The two-element core plays several roles at once.

It is a primitive algebraic law.

It is the target of each binary lawful view.

It is the factor used in subdirect representation.

It is the truth-value algebra for semilattice equations.

It is also the smallest structure capable of distinguishing satisfaction from nonsatisfaction while preserving meet.

The fixed-core dimension relative to $\mathbf{2}$is written

$$\nu_2(A).$$

It is the least number of two-element semilattice factors required in a separating subdirect embedding

$$A \hookrightarrow \mathbf{2}^m.$$

The subdirect requirement ensures that every coordinate takes both values. A constant map would provide no genuine two-state perspective and would not count as a full core view.

The choice of $\mathbf{2}$is not arbitrary. Every finite semilattice admits natural representation through two-state homomorphisms under suitable residual conditions. The semilattice operation also converts equations into transparent combinatorial statements about supports. This permits exact formulas that would be difficult to obtain immediately in richer varieties.

The two-state semilattice core therefore functions as a minimal laboratory for mathematical actualization. It is simple enough for complete analysis but rich enough to exhibit severe reconstruction overhead.

4.2 Reverse Inclusion and the Meet Convention

The free semilattice on generators

$$x_1, \dots, x_g$$

can be represented by the nonempty subsets of

$$[g] = \{1, \dots, g\}.$$

The subset

$$S \subseteq [g]$$

represents the term

$$x_S = \bigwedge_{i \in S} x_i.$$

Under this representation,

$$x_S \wedge x_T = x_{S \cup T}.$$

Thus the algebraic operation corresponds to set union.

Under ordinary inclusion,

$$S \subseteq T,$$

union is a join. Yet the operation is written as meet. This apparent mismatch is resolved by reversing the order.

Define

$$S \leq T \Leftrightarrow S \supseteq T.$$

Under reverse inclusion, a larger support set represents a lower semilattice element. The union $S \cup T$is then the greatest lower bound of $S$and T. Hence it is correctly interpreted as meet.

This convention should be kept explicit:

$$S \wedge T = S \cup T$$

because the semilattice order is reverse inclusion.

The generator $x_i$corresponds to the singleton support

$$\{i\}.$$

A meet of more generators corresponds to a larger support set and therefore to a lower element.

The full support

$$[g]$$

is the least element of the free semilattice under reverse inclusion. The empty support would represent an empty meet or greatest element, but it is absent when the signature contains no nullary identity constant. The free semilattice in the present setting therefore contains

$$2^g - 1$$

nonempty supports.

This convention is not merely cosmetic. It affects the interpretation of filters, ideals, irreducibility, and homomorphism supports. When set language and semilattice language are mixed, the order reversal must be remembered.

A coordinate set that is downward closed under ordinary inclusion may be upward closed in the reverse-inclusion semilattice order. A union-closed set family is meet-closed in the semilattice sense. A maximal support may be a minimal semilattice element.

Throughout the paper, the operation remains written as meet because the algebra belongs to the meet-semilattice variety. Set union is the concrete realization of that meet.

4.3 Free Semilattices as Nonempty Subset Systems

Let

$$F_g = F_{\mathrm{SL}}(g)$$

denote the free semilattice on $g$generators. Its carrier is

$$| F_g | = \{S \subseteq [g] : S \neq \emptyset\}.$$

Its operation is

$$S \wedge T = S \cup T.$$

Every nonempty subset appears exactly once because associativity, commutativity, and idempotence reduce every semilattice term to the set of variables occurring in it.

The generator set is

$$\{\{1\}, \ldots, \{g\}\}.$$

Every support $S$is generated by the singletons it contains:

$$S = \bigwedge_{i \in S} \{i\} = \bigcup_{i \in S} \{i\}.$$

The generator rank of $F_g$is exactly g. No singleton generator can be omitted because union cannot remove an index once it appears. If $\{i\}$is absent from a generating set, no generated support can equal $\{i\}$.

The free carrier size is

$$| F_g |= 2^g - 1.$$

The ordinary binary dimension of the carrier is therefore

$$\lceil \log_2(2^g - 1) \rceil = g.$$

The fixed-core view number is also g. For each generator index i, define

$$h_i(S) = \begin{cases} 1, & i \notin S, \\ 0, & i \in S, \end{cases}$$

or equivalently use the complemented convention

$$\tilde{h}_i(S) = \begin{cases} 1, & i \in S, \\ 0, & i \notin S, \end{cases}$$

depending on the chosen orientation of the two-element core and support filters. What matters is that the family of coordinate evaluations records generator membership in a homomorphism-compatible way.

A suitable g-coordinate family separates all distinct supports because two distinct subsets differ in at least one generator index.

Thus

$$\nu_2(F_g) = g.$$

The free semilattice has complexity profile

$$\mathcal{C}_2(F_g) = (g, 0, g).$$

It is simultaneously maximally uncommitted at generator rank g, actualized by no additional equation, and reconstructible at the information-theoretic minimum.

This free case serves as the baseline against which quotient actualization is measured. Any increase in $\nu_2$after imposing equations cannot be attributed simply to larger carrier size, because quotienting never increases the number of elements. It reflects a change in the geometry of admissible homomorphisms.

4.4 Homomorphisms as Admissible Binary Views

A map

$$h: F_g \to \mathbf{2}$$

is determined by its values on the free generators. Assigning a bit vector

$$a = (a_1, \ldots, a_g) \in \{0,1\}^g$$

determines the unique homomorphism

$$h_a(S) = \bigwedge_{i \in S} a_i.$$

Thus

$$h_a(S) = 1$$

exactly when every generator indexed by S receives value 1.

Let

$$C(a) = \{i \in [g]: a_i = 1\}.$$

Then

$$h_a(S) = 1 \Leftrightarrow S \subseteq C(a).$$

Every free homomorphism is therefore indexed by a subset

$$C \subseteq [g],$$

with

$$h_C(S) = 1 \Leftrightarrow S \subseteq C.$$

The set C may be called the support of the coordinate assignment.

The zero assignment $C = \emptyset$gives the constant-zero homomorphism. The full assignment $C = [g]$gives the constant-one homomorphism. Neither is surjective. Every nonempty proper subset Cyields a surjective homomorphism on the free semilattice when both values occur among the carrier elements.

For a quotient algebra

$$A = F_g/\theta,$$

not every free homomorphism descends to A. A homomorphism $h_C$descends precisely when it is constant on every θ-class. Equivalently, it must satisfy every defining equation of the quotient.

Thus admissible views of the actualized world correspond to points

$$C \subseteq [g]$$

satisfying the relation system.

The relation system does not merely identify algebra elements. It deletes or constrains potential homomorphic coordinates.

This is the mechanism by which quotienting can increase view number. The free world admits all bit assignments. An actualizing equation may eliminate many coordinates or force them into a structured family. Although fewer algebra elements remain, the surviving coordinates may separate them less efficiently.

The lawful-view problem becomes combinatorial:

Given the family of admissible supports $C \subseteq [g]$, what is the smallest subfamily whose evaluation maps separate all elements of the quotient?

For finite semilattice coordinate algebras, this minimum is governed by the join-irreducible structure of the admissible support family.

4.5 Equation Systems as Restrictions on Coordinate Supports

Consider a semilattice equation

$$x_P = x_Q,$$

where

$$P, Q \subseteq [g]$$

are nonempty support sets.

A coordinate assignment $C \subseteq [g]$satisfies this equation exactly when

$$h_C(P) = h_C(Q).$$

Using

$$h_C(S) = 1 \Leftrightarrow S \subseteq C,$$

the condition becomes

$$P \subseteq C \Leftrightarrow Q \subseteq C.$$

Thus each equation imposes a biconditional restriction on admissible coordinate supports.

For a system

$$R = \{P_1 = Q_1, \dots, P_s = Q_s\},$$

the solution family is

$$X_R = \{C \subseteq [g] : P_i \subseteq C \Leftrightarrow Q_i \subseteq C \text{ for all } i\}.$$

This is the set of all two-state assignments satisfying the relations.

Every point $C \in X_R$gives a homomorphism from the quotient coordinate algebra to $\mathbf{2}$. Conversely, every homomorphism arises from such a point.

The geometry of $X_R$therefore determines the lawful-view structure of the quotient.

A single equation excludes assignments for which exactly one side is contained in C. It divides the Boolean cube into three admissible regions:

- neither side is contained;
- both sides are contained;
- the containment conditions coincide through overlap.

With multiple equations, the admissible family becomes the intersection of these biconditional conditions.

The relations may mention many variables. Equation count and equation length are distinct. One relation can involve all $g$generators and produce a global dependency. The relation–view degree theorem bounds the view number using the number $s$of equations, but it does not claim that all one-equation presentations have low syntactic description length.

The crucial structural fact is that $X_R$is closed under union. This closure converts the solution family itself into a finite semilattice and makes join-irreducible analysis possible.

4.6 Union-Closed Solution Families

Let

$$C, D \in X_R.$$

For each defining equation

$$P_i = Q_i,$$

we have

$$P_i \subseteq C \Leftrightarrow Q_i \subseteq C$$

and

$$P_i \subseteq D \Leftrightarrow Q_i \subseteq D.$$

Now consider $C \cup D$. A support $P_i$is contained in $C \cup D$precisely when all its elements occur across the union. Because the equation conditions are preserved under the semilattice operation in the coordinate algebra, the pointwise meet of the two assignments is again a solution. Under the chosen support convention, this corresponds to union closure.

Hence

$$C, D \in X_R \Longrightarrow C \cup D \in X_R.$$

Therefore $X_R$is a union-closed family.

It contains the full support $[g]$, because both sides of every equation are contained in $[g]$. Depending on the treatment of constant maps and empty supports, it may also contain $\emptyset$.

The family X_R, ordered by inclusion and equipped with union, is a finite join-semilattice. Under reverse inclusion it may again be viewed as a meet-semilattice.

The coordinate algebra of the equation system can be understood through the functions

$$\hat{S}: X_R \to \mathbf{2}, \hat{S}(C) = h_C(S).$$

Two supports $S, T \subseteq [g]$represent the same quotient element exactly when

$$\hat{S} = \hat{T}$$

on all of X_R.

Thus the quotient world is the algebra of distinct coordinate functions induced by free terms on the solution family.

This is the equations–points duality in concrete finite form:

relations determine X_R,
X_R determines term-function identifications,
those identifications determine the quotient algebra.

The view number is then related to the minimum subfamily of $X_R$on which all distinct coordinate functions remain distinct.

The full solution family may contain many redundant points. Some points are unions of others and contribute no independent separation power. The indispensable structure is carried by join-irreducible solutions.

4.7 Join-Irreducible Solutions as Indispensable Views

Let Xbe a finite union-closed family. A nonminimal element

$$J \in X$$

is join-irreducible when

$$J = C \cup D$$

with $C, D \in X$implies

$$J = C \text{or} J = D.$$

Equivalently, Jcannot be assembled as the union of two strictly smaller admissible solutions.

Join-irreducible points are the atomic building blocks of the union-closed family. Every element of a finite join-semilattice is a union of join-irreducible elements lying below it.

In the present setting, each point

$$J \in X_R$$

corresponds to a homomorphism

$$h_J : A \to \mathbf{2}.$$

A join-reducible point often yields a coordinate whose effect is determined by the coordinates associated with smaller points. A join-irreducible point carries a separation pattern that cannot be reconstructed from proper admissible subpoints.

The minimum fixed-core view number satisfies

$$\nu_2(A) = |\, \mathrm{Jirr}(X_R) \,|$$

for the residual semilattice coordinate algebras considered here, after excluding trivial constant points in the appropriate convention.

This equality expresses the fixed-core dimension as a combinatorial count.

The upper bound in the relation–view degree theorem therefore reduces to bounding the possible sizes of join-irreducible solutions.

The key lemma states that if $X_R$is defined by $s$semilattice equations, then every join-irreducible solution has cardinality at most

$$s + 1.$$

Once this is known,

$$| \operatorname{Jirr}(X_R) | \leq \sum_{j=1}^{s+1} \binom{g}{j},$$

because there are only that many nonempty subsets of $[g]$of size at most $s + 1$.

The degree law follows immediately.

The significance of join-irreducibility is both structural and interpretive. A join-irreducible solution is an indispensable lawful point. It cannot be decomposed into smaller admissible points without losing a private aspect of the world's separation geometry.

The actualizing equations therefore bound how large an indispensable perspective can be. With $s$equations, no indispensable support needs more than $s + 1$generators. Yet the number of such supports can still grow as a polynomial of degree $s + 1$.

4.8 From Coordinate Families to Subdirect Dimension

Let

$$X_R \subseteq \mathbf{2}^g$$

be the solution set of the defining equations, identified with a family of supports. The coordinate algebra is

$$A_R = F_2(g)/\operatorname{Rad}_2(X_R).$$

Each point

$$C \in X_R$$

defines an evaluation homomorphism

$$\varepsilon_C \colon A_R \to \mathbf{2}.$$

The full evaluation map is

$$\varepsilon \colon A_R \to \mathbf{2}^{X_R},$$
$$\varepsilon(a) = (\varepsilon_C(a))_{C \in X_R}.$$

By construction of the radical congruence, this map is injective. Thus A_R embeds into a power of $\mathbf{2}$.

A subfamily

$$Y \subseteq X_R$$

still reconstructs A_R exactly when the restricted evaluation map

$$\varepsilon_Y : A_R \to \mathbf{2}^Y$$

remains injective.

The fixed-core dimension is therefore

$$\nu_2(A_R) = \min\{|\ Y\ | : Y \subseteq X_R,\ \varepsilon_Y \text{ injective}\}.$$

For the union-closed systems studied here, a canonical irredundant basis is supplied by the join-irreducible points. Every admissible point is generated by unions of these basis points, while each join-irreducible point has a private separation obligation.

Thus

$$A_R \hookrightarrow \mathbf{2}^{\mathrm{Jirr}(X_R)}$$

subdirectly, and no smaller set of coordinates suffices.

The passage from equations to fixed-core dimension can therefore be summarized as

$$R \longmapsto X_R \longmapsto \mathrm{Jirr}(X_R) \longmapsto \nu_2(A_R).$$

This chain is the computational and conceptual heart of the paper.

The equation system restricts possible points.

The union-closed point family determines the coordinate algebra.

The join-irreducible points determine the indispensable views.

Their count is the fixed-core subdirect dimension.

The relation–view theorem controls the final count by controlling the possible size of each join-irreducible point.

4.9 Why the Semilattice Core Is the Natural First Model

The semilattice core is not claimed to model every form of mathematical or physical actualization. It is the natural first model because it isolates combination, idempotence, and order in their simplest nontrivial form.

Semilattice law contains no inverse operation, no subtraction, no orientation, and no metric. It expresses only associative, commutative, idempotent combination. This makes its free objects combinatorially transparent.

Terms correspond to subsets.

Operation corresponds to union.

Equations become containment biconditionals.

Solution sets become union-closed families.

Indispensable views become join-irreducible points.

This transparency permits exact theorem formation.

The semilattice setting also captures a general feature of concept formation. Combining two descriptions or constraint sets may yield a stronger, lower, or more specific object. Repeating the same description adds nothing. Order of combination is irrelevant. Grouping is irrelevant. These are precisely the semilattice laws.

In this sense, semilattices model accumulation of conditions, conjunction of features, intersection of admissibility, or aggregation of information.

The two-element semilattice core then acts as a yes–no evaluator of whether a support lies within an admissible coordinate assignment. Each homomorphism records one lawful binary resolution of the larger world.

The model is also sufficiently weak to reveal representation overhead. More expressive cores may encode distinctions efficiently through richer term operations. The semilattice core cannot negate, subtract, or freely recombine coordinates. Its lawful views are constrained enough to expose the difference between ordinary bits and homomorphic bits.

At the same time, semilattices are classical enough that novelty claims can be sharply bounded. The contribution lies not in inventing free semilattices or two-element factor representations, but in deriving a quantitative relation among:

generator rank,
minimum relation count,

join-irreducible solution size,

and

fixed-core subdirect dimension.

The semilattice model therefore serves as a proof laboratory for a broader mathematics of actualization.

The later challenge is to determine which parts of the degree law survive in asymmetric difference cores, in richer signatures, and in algebraic systems with inverses or higher-arity operations. The semilattice case supplies the first complete benchmark.

5. The Relation–View Degree Theorem

5.1 Statement of the General Bound

Let Abe a finite semilattice coordinate algebra over the two-element semilattice core $\mathbf{2}$. Suppose that Ahas generator rank

$$\gamma(A) = g$$

and admits a presentation at that generator rank using at most $s$additional semilattice equations.

Then the fixed-core subdirect dimension satisfies

$$\boxed{\nu_2(A) \leq \sum_{j=1}^{s+1} \binom{g}{j}.}$$

Equivalently, no more than the first $s + 1$nonzero levels of the g-dimensional Boolean cube are required to index the indispensable two-state homomorphisms of A.

If the presentation uses the minimum possible number of equations, so that

$$\sigma_2(A) = s,$$

then

$$\boxed{\nu_2(A) \leq \sum_{j=1}^{\sigma_2(A)+1} \binom{\gamma(A)}{j}.}$$

This is the Relation–View Degree Theorem.

The theorem connects three quantities that are usually studied separately:

$$\gamma(A), \sigma_2(A), \nu_2(A).$$

Generator rank determines the size of the ambient coordinate cube. Actualization number determines how many levels of that cube can contain indispensable lawful views. Fixed-core dimension counts how many such views may be required.

For fixed s, the dominant term is

$$\binom{g}{s+1}.$$

Therefore,

$$\nu_2(A) = O(g^{s+1}).$$

The number of actualizing equations controls the maximum polynomial degree of lawful reconstruction complexity.

The theorem does not merely state that a finite presentation gives some finite representation. It supplies an explicit quantitative bound. It says that $s$additional identities restrict indispensable two-state views to supports involving at most $s+1$generators.

The proof proceeds through the solution geometry of the equations. The equations determine a union-closed family of zero sets. The indispensable views are exactly the nontrivial join-irreducible members of that family. A sparsity argument then proves that each such member has cardinality at most $s+1$.

The central chain is

$$s \text{ equations} \Longrightarrow \text{join-irreducible supports of size at most } s+1$$

$$\Longrightarrow \text{ at most } \sum_{j=1}^{s+1} \binom{g}{j} \text{ indispensable views.}$$

This is the mathematical mechanism behind the Divine Spark complexity law.

5.2 Equation Systems on g Generators

Let

$$F_g$$

be the free semilattice on generators

$$x_1, \dots, x_g.$$

Every term is represented by a nonempty support

$$U \subseteq [g],$$

with

$$x_U = \bigwedge_{i \in U} x_i.$$

Consider a system of $s$equations

$$R = \{U_1 = V_1, \dots, U_s = V_s\},$$

where each $U_i$and $V_i$is a nonempty subset of $[g]$.

A two-valued assignment to the generators is a vector

$$a = (a_1, \dots, a_g) \in \{0,1\}^g.$$

For the proof, it is convenient to encode this assignment by its zero set

$$Z(a) = \{i \in [g] : a_i = 0\}.$$

The value of a term x_U under the assignment is

$$x_U(a) = 1 \Leftrightarrow U \cap Z(a) = \emptyset.$$

Thus x_U evaluates to 1 exactly when none of its variables has been assigned 0.

The equation

$$U_i = V_i$$

is satisfied by the zero set Z exactly when

$$U_i \cap Z = \emptyset \Leftrightarrow V_i \cap Z = \emptyset.$$

Equivalently,

$$Z \cap U_i \neq \emptyset \Leftrightarrow Z \cap V_i \neq \emptyset.$$

The zero set must hit the two sides together or miss the two sides together.

Define the complete solution family

$$X_R = \{Z \subseteq [g] : Z \cap U_i \neq \emptyset \Leftrightarrow Z \cap V_i \neq \emptyset \text{ for every } i\}.$$

Every $Z \in X_R$ determines a homomorphism

$$h_Z : A_R \to \mathbf{2}$$

from the coordinate algebra of the equation system.

The empty zero set gives the constant-one assignment. The full zero set gives the constant-zero assignment. These constant maps are not surjective and are excluded when counting genuine two-state factors. Nonempty proper members of X_R correspond to surjective views when both truth values occur on the coordinate algebra.

The family X_R is closed under union. To see this, let

$$Z, W \in X_R.$$

Suppose $Z \cup W$ hits U_i. Then either Z or W hits U_i. Since that set satisfies the equation, it also hits V_i. Therefore $Z \cup W$ hits V_i. The converse follows symmetrically.

Hence

$$Z, W \in X_R \Longrightarrow Z \cup W \in X_R.$$

The solution family is therefore a finite join-semilattice under union.

The quotient algebra determined by $R$is recovered from equality of term evaluations on X_R. Two free terms $x_U$and $x_V$represent the same element exactly when

$$x_U(Z) = x_V(Z)$$

for every $Z \in X_R$.

Thus the equation system determines a point family, and the point family determines the quotient:

$$R \longmapsto X_R \longmapsto A_R.$$

The fixed-core representation problem becomes the problem of finding the smallest subfamily of $X_R$that preserves all distinctions among the coordinate functions of A_R.

5.3 The Bound $v_2(A) \leq \Sigma$ from j = 1 to s + 1 of C(g,j)

Let

$$\mathrm{Jirr}(X_R)$$

denote the join-irreducible members of the union-closed solution family.

A nonempty member $Z \in X_R$is join-irreducible when

$$Z = Z_1 \cup Z_2, Z_1, Z_2 \in X_R,$$

implies

$$Z = Z_1 \text{or} Z = Z_2.$$

The join-irreducible solutions are precisely the points that cannot be assembled from two strictly smaller solutions.

For the finite coordinate algebras considered here, the indispensable surjective coordinates correspond exactly to the nonempty proper join-irreducible members of X_R. Therefore,

$$\nu_2(A_R) = |\{Z \in \mathrm{Jirr}(X_R) : \emptyset \neq Z \neq [g]\}|.$$

The crucial sparsity result is

$$Z \in \mathrm{Jirr}(X_R) \Longrightarrow |Z| \leq s + 1.$$

Once this has been proved, the dimension bound follows by elementary counting. There are

$$\binom{g}{j}$$

subsets of $[g]$having cardinality j. Therefore the total number of possible nonempty join-irreducible supports is at most

$$\binom{g}{1} + \binom{g}{2} + \cdots + \binom{g}{s+1}.$$

Hence

$$\nu_2(A_R) \leq \sum_{j=1}^{s+1} \binom{g}{j}.$$

The estimate may count subsets that are not solutions, subsets that are solutions but join-reducible, and the full zero set if its cardinality lies in the range. It is therefore an upper bound rather than an assertion that every low-cardinality support is indispensable.

The power of the theorem lies in its independence from equation length. Each defining equation may involve many or all $g$generators. Nevertheless, $s$equations cannot produce an indispensable zero support containing more than $s + 1$elements.

The equations may be globally written, but each irreducible lawful view possesses a sparse certificate.

This local sparsity forces the global polynomial bound.

5.4 Why Join-Irreducible Solutions Have Size at Most s + 1

Let

$$Z \in X_R$$

be join-irreducible.

Form the union of all proper solution subsets of Z:

$$Z^- = \bigcup\{W \in X_R : W \subsetneq Z\}.$$

Because $X_R$is finite and union closed,

$$Z^- \in X_R.$$

If

$$Z^- = Z,$$

then $Z$is the union of proper solution subsets. Since the family is finite, Zcan then be expressed as a union of finitely many proper members. Grouping those members into two unions would express Zas the union of two strictly smaller solutions, contradicting join-irreducibility.

Therefore,

$$Z^- \subsetneq Z.$$

Choose

$$z_0 \in Z \setminus Z^-.$$

The element $z_0$belongs to no proper solution subset of Z.

Consequently, $Z$is inclusion-minimal among the solutions contained in Zthat contain z_0. Indeed, if

$$W \in X_R, z_0 \in W \subsetneq Z,$$

then $z_0$would belong to a proper solution subset of Z, contrary to its choice.

Now begin with

$$W_0 = \{z_0\}.$$

The singleton W_0need not satisfy the equation system. If it does, minimality immediately gives

$$Z = W_0$$

and $| Z | = 1$.

Otherwise, W_0violates at least one equation. Suppose that at some stage $W_t \subseteq Z$violates

$$U_i = V_i.$$

After exchanging the two sides if necessary, assume

$$W_t \cap U_i \neq \emptyset$$

but

$$W_t \cap V_i = \emptyset.$$

Because

$$W_t \subseteq Z$$

and $Z$is a valid solution, Zhits both sides whenever it hits one. Therefore,

$$Z \cap V_i \neq \emptyset.$$

Choose

$$z_{t+1} \in Z \cap V_i$$

and set

$$W_{t+1} = W_t \cup \{z_{t+1}\}.$$

The selected equation now has both sides hit.

Once both sides of an equation have been hit, later additions cannot make either side unhit. Thus that equation can never again become violated.

Every repair therefore permanently resolves at least one equation not previously repaired.

Since there are only $s$equations, at most $s$repairs are possible. Starting with one root element, the procedure produces a valid solution after at most $s$additions:

$$| W | \leq s + 1.$$

The final Wsatisfies

$$z_0 \in W \subseteq Z.$$

By the minimality of Zamong solutions contained in $Z$and containing z_0,

$$W = Z.$$

Therefore,

$$\boxed{| Z | \leq s + 1.}$$

This is the Sparsity Lemma.

The proof gives more than a cardinality estimate. It constructs every join-irreducible solution from one distinguished element plus at most one repair element per defining equation.

5.5 Dependency Certificates for Join-Irreducibility

The repair proof yields a finite certificate describing how an indispensable view depends on the defining relations.

For a join-irreducible solution Z, choose the root

$$z_0 \in Z$$

that belongs to no proper solution subset of Z. Then construct a sequence

$$W_0 \subset W_1 \subset \cdots \subset W_t = Z,$$

where

$$W_0 = \{z_0\},$$

and every extension has the form

$$W_{r+1} = W_r \cup \{z_{r+1}\}.$$

The new element z_{r+1}repairs a violated equation indexed by some

$$i_r \in \{1, \dots, s\}.$$

The certificate may be recorded as

$$\mathcal{D}(Z) = (z_0; (i_0, z_1), (i_1, z_2), \dots, (i_{t-1}, z_t)).$$

The equation indices appearing in the certificate are distinct, because an equation that has once been repaired by making both sides hit cannot later become violated.

Hence

$$t \leq s.$$

The support Zhas

$$| Z | = t + 1 \leq s + 1.$$

The certificate can be interpreted as a dependency history. The root element initiates the support. Each defining relation forces the inclusion of at most one additional element needed to restore consistency. The final join-irreducible view is therefore connected to the relation system through a chain of at most $s$forced repairs.

This does not mean that each join-irreducible has a unique certificate. Different choices of violated equation or repair element may lead to different sequences. What is invariant is the existence of at least one certificate of length at most s.

These certificates are useful computationally. Instead of merely reporting that a point is join-irreducible, an audit program can output:

- the root element;
- the order in which relations are activated;
- the side of each relation first hit;
- the repair element chosen from the opposite side;
- the final support.

Such a certificate is short enough to inspect directly.

The certificate also explains the degree $s + 1$. The indispensable support contains one initial element and at most one new dependency witness per equation.

The theorem is therefore not a mysterious binomial coincidence. The binomial level arises from the maximum size of a relation-dependency certificate.

5.6 Private Witnesses and Irredundant Coordinates

To prove that every nontrivial join-irreducible solution is indispensable, one must show that it performs a separation task not performed by the remaining solution points.

Let

$$Z \in X_R$$

be a nonempty proper join-irreducible solution.

Again define

$$Z^- = \bigcup\{W \in X_R : W \subsetneq Z\}.$$

Choose

$$a \in Z \setminus Z^-.$$

Let

$$U = [g] \setminus Z$$

and

$$V = U \cup \{a\}.$$

Because $Z \neq [g]$, the support $U$is nonempty and therefore represents a valid free semilattice term.

Evaluate the two terms $x_U$and $x_V$at the zero set Z.

Since

$$U \cap Z = \emptyset,$$

we have

$$x_U(Z) = 1.$$

Since

$$a \in V \cap Z,$$

we have

$$x_V(Z) = 0.$$

Thus Zseparates the pair

$$x_U, x_V.$$

Now let

$$W \in X_R, W \neq Z.$$

If

$$a \notin W,$$

then $U$and Vhave the same intersection with W, because their only difference is the element a. Hence

$$x_U(W) = x_V(W).$$

If

$$a \in W,$$

then Wcannot be a proper subset of Z, because $a$belongs to no proper solution subset of Z. Therefore Wcontains some element outside Z. Every element outside Zlies in U, so

$$W \cap U \neq \emptyset.$$

Then both $U$and Vmeet W, and

$$x_U(W) = x_V(W) = 0.$$

Therefore $Z$is the only solution point distinguishing $x_U$from x_V.

The pair

$$(x_U, x_V)$$

is a private witness for the coordinate Z.

Removing $Z$from the evaluation family identifies these two quotient elements. Hence the coordinate is indispensable.

Conversely, if

$$Z = Z_1 \cup Z_2$$

with $Z_1, Z_2 \subsetneq Z$both solutions, then evaluation at $Z$is determined by evaluations at $Z_1$and Z_2. In zero-set form,

$$x_U(Z) = 1$$

exactly when

$$U \cap Z_1 = \emptyset \text{and} U \cap Z_2 = \emptyset.$$

Thus

$$x_U(Z) = x_U(Z_1) \wedge x_U(Z_2).$$

The coordinate $Z$is a pointwise meet of the two smaller coordinates and is redundant.

Hence:

$$\boxed{\text{indispensable lawful views} = \text{nonempty proper join-irreducible solutions.}}$$

This establishes the exact bridge between solution-family combinatorics and fixed-core subdirect dimension.

5.7 Proof of the Fixed-Core Dimension Bound

The full theorem can now be proved in a compact sequence.

Let Abe a g-generated residual semilattice coordinate algebra presented by $s$equations over $\mathbf{2}$.

Choose a presentation

$$R = \{U_1 = V_1, \dots, U_s = V_s\}$$

on the minimum generator set.

Let $X_R$be the family of zero-set solutions.

First, $X_R$is union closed.

Second, every nonempty proper join-irreducible member $Z$of $X_R$gives an indispensable surjective homomorphism

$$h_Z : A \to \mathbf{2}.$$

Every join-reducible point is redundant because its evaluation is the meet of evaluations at smaller solution points.

Therefore,

$$\nu_2(A) = |\ \mathrm{Jirr}^*(X_R)\ |,$$

where $\mathrm{Jirr}^*(X_R)$denotes the nonempty proper join-irreducible members.

Third, the Sparsity Lemma gives

$$|\ Z\ | \leq s + 1$$

for every $Z \in \mathrm{Jirr}^*(X_R)$.

Fourth, there are only

$$\sum_{j=1}^{s+1} \binom{g}{j}$$

nonempty subsets of $[g]$of size at most $s + 1$.

Consequently,

$$\nu_2(A) \leq \sum_{j=1}^{s+1} \binom{g}{j}.$$

If the presentation uses the minimum possible number of equations at generator rank g, then

$$s = \sigma_2(A),$$

and therefore

$$\boxed{\nu_2(A) \leq \sum_{j=1}^{\sigma_2(A)+1} \binom{\gamma(A)}{j}.}$$

No enumeration of quotient multiplication tables is required.

No assumption is made that the equations have bounded length.

No assumption is made that the quotient carrier is small.

The bound follows solely from:

- the number of generators;
- the number of defining equations;
- the union closure of the two-state solution family;
- and the sparsity of join-irreducible solutions.

The theorem is therefore structural rather than merely computational.

The computational deposit verifies the result on exhaustive small systems, but the proof applies to all finite residual semilattice coordinate algebras in the stated setting.

5.8 The Zero-Relation Linear Case

Let

$$s = 0.$$

No additional relation is imposed, so the coordinate algebra is the free semilattice

$$F_g.$$

Every zero set

$$Z \subseteq [g]$$

is admissible. Thus

$$X_\emptyset = 2^{[g]}.$$

Under union, the nonempty join-irreducible members of the Boolean family are exactly the singleton sets:

$$\{1\}, \ldots, \{g\}.$$

Every larger set is a union of its singleton subsets.

Therefore,

$$\nu_2(F_g) = g.$$

The general theorem gives

$$\nu_2(F_g) \leq \binom{g}{1} = g,$$

so equality holds.

The free world is reconstructed by one coordinate for each generator. It has no fixed-core representation overhead:

$$\nu_2(F_g) = \lceil \log_2 | F_g | \rceil = g,$$

because

$$| F_g | = 2^g - 1.$$

The zero-relation case establishes the base of the hierarchy:

$$\sigma_2(A) = 0 \Longrightarrow \nu_2(A) \leq \gamma(A).$$

In the relatively free case, lawful reconstruction is linear in primitive generator rank.

The introduction of even one actualizing relation changes the possible degree.

5.9 The One-Relation Quadratic Case

Let

$$s = 1.$$

The Relation–View Degree Theorem gives

$$\nu_2(A) \leq \binom{g}{1} + \binom{g}{2}.$$

Therefore,

$$\nu_2(A) \leq g + \frac{g(g-1)}{2} = \frac{g(g+1)}{2}.$$

Every indispensable solution support has size at most two. The only possible irreducible lawful views are therefore:

- singleton zero sets;
- two-element zero sets.

A single equation can create pairwise dependencies among generators, but it cannot force a genuinely irreducible support involving three or more generators.

Hence one actualizing relation may increase view complexity from linear to quadratic, but never to cubic.

The general binomial estimate is not the exact extremal value. A more detailed four-block analysis of a single equation gives

$$\nu_2 = r + pq + c(p + q + 1),$$

where:

- $c$is the number of generators common to both sides;
- $p$is the number occurring only on the left;
- $q$is the number occurring only on the right;
- $r$is the number occurring on neither side.

Maximization over all one-equation presentations yields

$$\boxed{M_2(g,1) = \left\lfloor \frac{g^2+g+1}{3} \right\rfloor.}$$

Thus the exact one-relation ceiling is approximately

$$\frac{g^2}{3},$$

rather than the coarser general estimate of approximately

$$\frac{g^2}{2}.$$

The one-relation world therefore exhibits a sharp but bounded amplification:

$$(g,0,g) \longrightarrow (g,1,\Theta(g^2)).$$

This is the first nontrivial level of the Divine Spark complexity hierarchy.

5.10 The General Degree-(s + 1) Hierarchy

For fixed s, the leading term in the upper bound is

$$\binom{g}{s+1} = \frac{g^{s+1}}{(s+1)!} + O(g^s).$$

Therefore,

$$\nu_2(A) = O(g^{s+1}).$$

The first levels are:

$$s = 0: \nu_2(A) = O(g),$$
$$s = 1: \nu_2(A) = O(g^2),$$
$$s = 2: \nu_2(A) = O(g^3),$$
$$s = 3: \nu_2(A) = O(g^4).$$

In general,

$$\boxed{\sigma_2(A) = s \Longrightarrow \nu_2(A) = O(g^{s+1})}$$

for fixed s.

The degree is sharp. The block-chain constructions developed in the next section satisfy

$$\nu_2(A) = \prod_{i=0}^{s} b_i,$$

where

$$b_0 + \cdots + b_s = g.$$

Balancing the block sizes gives

$$b_i \approx \frac{g}{s+1},$$

and hence

$$\nu_2(A) \approx \left(\frac{g}{s+1}\right)^{s+1}.$$

Therefore,

$$\nu_2(A) = \Omega(g^{s+1}).$$

Combining the bounds gives

$$\boxed{M_2(g, s) = \Theta(g^{s+1})}$$

for every fixed s.

This hierarchy gives an exact asymptotic meaning to the complexity of actualization:

each additional defining relation can raise the maximum polynomial degree by one.

The result does not say that every new equation raises the degree. Many equations are redundant, weak, or poorly arranged. It says that $s$equations permit no degree greater than $s + 1$, and that appropriately organized systems can attain that degree.

The hierarchy also establishes a limitation. No bounded number of actualizing relations can force view complexity exponential in g.

To reach

$$\nu_2(A) = 2^{\Theta(g)},$$

the relation count must itself grow with g.

Thus the Divine Spark complexity law has both an amplification principle and a conservation principle.

A small relational basis may generate consequences vastly larger than itself.

But the asymptotic scale of those consequences remains bounded by the formal complexity of the actualizing basis.

6. Sharp Block-Chain Constructions

6.1 Partitioning the Generator Set into s + 1 Blocks

The Relation–View Degree Theorem gives the general upper bound

$$\nu_2(A) \leq \sum_{j=1}^{s+1} \binom{g}{j}$$

for a g-generated semilattice coordinate world presented by $s$equations. To show that the resulting degree $s + 1$is unavoidable, one needs explicit families whose view number grows as

$$g^{s+1}.$$

The required construction begins by partitioning the generator set into $s + 1$nonempty disjoint blocks:

$$[g] = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_s.$$

Write

$$| V_i | = b_i,$$

so that

$$b_0 + b_1 + \cdots + b_s = g.$$

Each block contains a local collection of primitive generators. The actualizing equations will not identify the generators individually. Instead, they identify the complete meets of adjacent blocks.

For each i, define

$$m_i = \bigwedge_{x_j \in V_i} x_j.$$

The terms $m_i$represent the total conjunction of all generators in block V_i.

The construction imposes a chain of equalities among these block meets. It therefore creates dependence among the blocks while preserving the internal individuality of the generators within each block.

The partition supplies the combinatorial source of the high view number. An indispensable lawful view will eventually correspond to choosing one generator from each block. The number of such transversals is

$$b_0 b_1 \cdots b_s.$$

When the blocks are balanced, this product has order

$$g^{s+1}.$$

Thus the construction translates $s+1$independently selectable blocks into a degree-$(s+1)$ family of indispensable perspectives.

6.2 The Chain of Actualizing Equations

Impose the $s$equations

$$\begin{aligned} m_0 &= m_1, \\ m_1 &= m_2, \\ &\vdots \\ m_{s-1} &= m_s. \end{aligned}$$

Equivalently,

$$\bigwedge_{x_j \in V_0} x_j = \bigwedge_{x_j \in V_1} x_j,$$
$$\bigwedge_{x_j \in V_1} x_j = \bigwedge_{x_j \in V_2} x_j,$$

and so on.

Let the resulting quotient algebra be denoted

$$A(V_0, \dots, V_s)$$

or, when only block sizes matter,

$$A(b_0, \dots, b_s).$$

The displayed relations imply

$$m_0 = m_1 = \cdots = m_s$$

in the quotient. Yet only $s$adjacent equalities are required to enforce equality of all $s+1$block meets.

To understand the lawful two-state views, encode an assignment by its zero set

$$Z \subseteq [g].$$

The block term $m_i$evaluates to zero precisely when Zmeets V_i:

$$m_i(Z) = 0 \Leftrightarrow Z \cap V_i \neq \emptyset.$$

The chain equations therefore require

$$Z \cap V_0 \neq \emptyset \Leftrightarrow Z \cap V_1 \neq \emptyset \Leftrightarrow \cdots \Leftrightarrow Z \cap V_s \neq \emptyset.$$

Because the blocks partition all generators, there are only two possibilities.

Either

$$Z = \emptyset,$$

so that no block is hit, or

$$Z \cap V_i \neq \emptyset$$

for every i.

Thus the solution family is

$$X = \{\emptyset\} \cup \{Z \subseteq [g] : Z \cap V_i \neq \emptyset \text{ for every } i\}.$$

The nonempty solutions are exactly the subsets that choose at least one element from every block.

This direct product-like structure is responsible for all of the exact formulas that follow.

6.3 Exact Carrier Formula

The quotient carrier may be counted by placing each free semilattice term into a canonical blockwise normal form.

A free term is represented by a nonempty support

$$S \subseteq [g].$$

Write its block components as

$$S_i = S \cap V_i.$$

The chain relations identify every support containing an entire block with the common saturated class generated by

$$m_0 = m_1 = \cdots = m_s.$$

For supports that do not exhaust any block, the quotient class is determined by the blockwise complements

$$C_i = V_i \setminus S_i.$$

Each $C_i$is then nonempty.

Conversely, a tuple

$$(C_0, \dots, C_s), \emptyset \neq C_i \subseteq V_i,$$

determines one quotient class, with the all-full tuple serving as the normal representative of the common saturated class.

The quotient classes are therefore indexed by choices of one nonempty subset from each block.

Block $V_i$has

$$2^{b_i} - 1$$

nonempty subsets. Hence

$$\boxed{| A(b_0, \dots, b_s) | = \prod_{i=0}^{s} (2^{b_i} - 1).}$$

This formula may also be recovered through the coordinate functions on the solution family. Distinct quotient elements correspond to distinct blockwise avoidance profiles, and each block contributes exactly $2^{b_i} - 1$possible nonempty profiles.

The carrier formula shows that the quotient remains extremely large. For balanced blocks of size approximately $g/(s+1)$,

$$| A | \approx \left(2^{g/(s+1)}\right)^{s+1} = 2^g.$$

More precisely, for fixed $s$and block sizes tending to infinity,

$$| A |= 2^g \prod_{i=0}^{s} (1 - 2^{-b_i}).$$

Therefore,

$$\frac{| A |}{2^g} \longrightarrow 1.$$

The chain relations alter the structure of the free world while preserving almost all of its exponential carrier scale.

This is an important feature of the construction. The high lawful-view number does not arise because the quotient has become small or degenerate. It remains close in cardinality to the largest possible g-generated semilattice.

6.4 Exact Fixed-Core View Formula

The nonempty solution family consists of all sets meeting every block:

$$X^* = \{Z \subseteq [g] : Z \cap V_i \neq \emptyset \text{ for every } i\}.$$

A member $Z \in X^*$is inclusion-minimal exactly when it contains one and only one element from each block.

Such a set has the form

$$T = \{v_0, \ldots, v_s\}, v_i \in V_i.$$

These sets are the transversals of the block partition.

Every transversal is join-irreducible in the union-closed family. It has no proper nonempty solution subset, because deleting any one of its elements leaves one block unhit.

Conversely, every solution containing two or more elements from some block is join-reducible. Suppose Zcontains distinct elements

$$u, v \in Z \cap V_i.$$

Choose, for every other block V_j, at least one element of $Z \cap V_j$. One can then construct two proper solution subsets of Z, one containing $u$but not v, and the other containing $v$but not u, whose union is Z. Thus $Z$is not join-irreducible.

Therefore the nontrivial join-irreducible solutions are exactly the block transversals.

The number of transversals is

$$b_0 b_1 \cdots b_s.$$

Since the join-irreducible solutions are precisely the indispensable two-state views,

$$\boxed{\nu_2(A(b_0, \ldots, b_s)) = \prod_{i=0}^{s} b_i .}$$

This is the exact fixed-core view formula.

The contrast with the carrier formula is striking:

$$| A | = \prod_{i=0}^{s} (2^{b_i} - 1),$$

while

$$\nu_2(A) = \prod_{i=0}^{s} b_i.$$

The carrier multiplies the numbers of nonempty subsets in the blocks. The view number multiplies the numbers of individual choices in the blocks.

Each indispensable view records one irreducible combination of local zeros, selecting exactly one generator from every block.

6.5 Identification of the Indispensable Views

Let

$$T = \{v_0, \ldots, v_s\}$$

be a transversal with $v_i \in V_i$.

The associated homomorphism

$$h_T : A \to \mathbf{2}$$

evaluates a quotient term according to whether its support avoids T. The transversal is a legal zero set because it meets every block.

To show that h_T is indispensable, construct a pair of quotient terms separated only by T.

Let

$$U = [g] \setminus T.$$

Choose one distinguished element, for example v_0, and define

$$W = U \cup \{v_0\}.$$

At the point T,

$$U \cap T = \emptyset,$$

so

$$x_U(T) = 1,$$

whereas

$$W \cap T = \{v_0\},$$

so

$$x_W(T) = 0.$$

Now consider another admissible nonempty solution $Z \neq T$.

If $v_0 \notin Z$, then U and W have the same intersection with Z, and the two terms agree.

If $v_0 \in Z$, then because $Z \neq T$ and Z meets every block, either Z contains another element of V_0 or differs from T in some other block. In either case, Z contains an element outside T, and therefore an element of U. Both U and W meet Z, so both terms evaluate to zero.

Thus Talone separates the pair

$$x_U, x_W.$$

Every transversal has a private witness pair. None can be removed from a separating family.

Conversely, every larger admissible zero set is a union of transversals contained within it. Its evaluation coordinate is determined by the corresponding transversal coordinates. It contributes no additional irreducible separation.

The set of all transversals therefore forms a canonical minimum subdirect representation:

$$A \hookrightarrow \mathbf{2}^{\,b_0 b_1 \cdots b_s}.$$

This representation is not merely minimum by counting. Each coordinate has an explicit algebraic reason for being indispensable.

6.6 Minimality of the Equation Basis

The construction is presented by $s$equations, so immediately

$$\sigma_2(A) \le s.$$

To prove equality, one must rule out a different presentation at generator rank $g$using fewer equations.

Suppose Acould be presented using at most $s - 1$equations. The Relation–View Degree Theorem would then imply

$$\nu_2(A) \le \sum_{j=1}^{s} \binom{g}{j}.$$

But the block-chain construction has

$$\nu_2(A) = \prod_{i=0}^{s} b_i.$$

Therefore, whenever

$$\prod_{i=0}^{s} b_i > \sum_{j=1}^{s} \binom{g}{j},$$

no presentation using $s-1$or fewer equations can exist at generator rank g.

Under this inequality,

$$\sigma_2(A) \geq s.$$

Combined with the displayed s-equation presentation,

$$\boxed{\sigma_2(A) = s.}$$

For fixed s, balanced block sizes eventually satisfy the inequality because

$$\prod_{i=0}^{s} b_i = \Theta(g^{s+1}),$$

whereas

$$\sum_{j=1}^{s} \binom{g}{j} = \Theta(g^s).$$

Thus the equation basis is minimal for all sufficiently large balanced constructions.

This argument is significant because it does not rely on syntactic independence of the displayed equations. It uses the resulting representation complexity to prove that no entirely different shorter presentation is possible.

The view number therefore certifies actualization complexity.

A large enough lawful-view requirement becomes a lower bound on the number of relations needed to produce the world.

6.7 Balanced Blocks and Extremal Growth

For fixed $g$and s, the view number is

$$\nu_2(A) = b_0 b_1 \cdots b_s$$

subject to

$$b_0 + \cdots + b_s = g, b_i \geq 1.$$

The product is maximized when the block sizes differ by at most one.

Write

$$g = (s+1)q + r,$$

where

$$0 \leq r < s + 1.$$

Then an optimal partition has:

$$r$$

blocks of size

$$q + 1,$$

and

$$s + 1 - r$$

blocks of size

$$q.$$

The maximum view number produced by the block-chain construction is therefore

$$\boxed{q^{\,s+1-r}(q+1)^r.}$$

In asymptotic form,

$$\nu_2(A) = \left(\frac{g}{s+1}\right)^{s+1}(1 + o(1))$$

for fixed s.

This follows from the discrete arithmetic–geometric mean principle. If two block sizes differ by at least two, transferring one generator from the larger block to the smaller increases their product:

$$(a-1)(b+1) - ab = a - b - 1 > 0$$

whenever $a \geq b + 2$.

Repeated balancing therefore maximizes the total product.

The balanced construction distributes actualizing dependence evenly across all $s + 1$blocks. No one block becomes a bottleneck. Every block contributes comparably to the number of irreducible transversals.

This balanced organization is the combinatorial source of extremal amplification.

6.8 Achieving Θ(g^(s + 1)) Lawful Views

The general theorem gives

$$\nu_2(A) \leq \sum_{j=1}^{s+1} \binom{g}{j}.$$

For fixed s,

$$\sum_{j=1}^{s+1} \binom{g}{j} = \frac{g^{s+1}}{(s+1)!} + O(g^s).$$

The balanced block-chain construction gives

$$\nu_2(A) = \left(\frac{g}{s+1}\right)^{s+1}(1 + o(1)).$$

Therefore,

$$\nu_2(A) = \Omega(g^{s+1}).$$

Combining the upper and lower bounds yields

$$\boxed{M_2(g,s) = \Theta(g^{s+1})}$$

for every fixed s.

The constants in the upper and lower bounds differ. The general theorem permits approximately

$$\frac{g^{s+1}}{(s+1)!}$$

views, while the balanced chain produces approximately

$$\frac{g^{s+1}}{(s+1)^{s+1}}.$$

Determining the exact leading constant for $s \geq 2$remains open.

Nevertheless, the degree is exact. No improved general theorem can lower the exponent below $s+1$, because the block construction attains that exponent.

The result establishes a full polynomial hierarchy:

$$s = 0 \Rightarrow \Theta(g),$$
$$s = 1 \Rightarrow \Theta(g^2),$$
$$s = 2 \Rightarrow \Theta(g^3),$$

and in general,

$$s \Rightarrow \Theta(g^{s+1}).$$

The number of defining relations behaves as a degree parameter for lawful reconstruction.

6.9 Why the Polynomial Degree Bound Is Sharp

A bound is sharp in degree when its exponent cannot be decreased uniformly.

Suppose one attempted to replace the Relation–View Degree Theorem by a stronger assertion of the form

$$\nu_2(A) = O(g^s)$$

for algebras presented by $s$equations.

The balanced block-chain family contradicts this immediately, because

$$\nu_2(A) = \Theta(g^{s+1}).$$

Similarly, no bound of the form

$$O(g^{s+1-\varepsilon})$$

with fixed $\varepsilon > 0$can hold.

The sharpness comes from the combinatorics of transversals. Each of the $s + 1$blocks contributes one independent choice. Multiplying those choices produces degree $s + 1$.

The relation chain has exactly enough equations to synchronize the blocks:

$$s \text{ equations} \Longrightarrow s + 1 \text{ mutually linked blocks.}$$

The indispensable views select one local witness from each linked block.

Thus the extra degree relative to equation count is not accidental. One begins with a root choice, and every relation can introduce one further independent dependency choice. This mirrors the $s + 1$support bound in the general proof.

The upper-bound proof and lower-bound construction therefore share the same underlying mechanism.

The proof says that one root plus $s$relation repairs limits every irreducible view to at most $s + 1$generators.

The construction arranges $s + 1$blocks so that every choice of one generator from each block becomes irreducible.

The theorem and construction meet at the same combinatorial depth.

6.10 Concrete Two-, Three-, and Five-Relation Worlds

Consider first two actualizing relations.

Partition fifteen generators into three blocks of five:

$$b_0 = b_1 = b_2 = 5.$$

Impose

$$m_0 = m_1, m_1 = m_2.$$

Then

$$\gamma(A) = 15,$$
$$| A | = (2^5 - 1)^3 = 31^3 = 29{,}791,$$

and

$$\nu_2(A) = 5^3 = 125.$$

The ordinary binary information bound is

$$\lceil \log_2 29{,}791 \rceil = 15.$$

Thus a world ordinarily labelable with fifteen bits requires 125 lawful two-state views.

Moreover,

$$125 > \binom{15}{1} + \binom{15}{2} = 120.$$

If the world were presentable by one equation, the Relation–View Degree Theorem would bound its view number by 120. Therefore,

$$\sigma_2(A) = 2.$$

The two displayed relations are genuinely necessary.

Now consider three relations.

Partition forty-four generators into four blocks of eleven:

$$b_0 = b_1 = b_2 = b_3 = 11.$$

Then

$$\gamma(A) = 44,$$
$$|\,A\,| = (2^{11} - 1)^4 = 2047^4 = 17{,}557{,}851{,}463{,}681,$$

and

$$\nu_2(A) = 11^4 = 14{,}641.$$

The ordinary binary dimension remains

$$44.$$

The upper bound for a two-relation presentation is

$$\binom{44}{1} + \binom{44}{2} + \binom{44}{3} = 44 + 946 + 13{,}244 = 14{,}234.$$

Since

$$14{,}641 > 14{,}234,$$

the algebra cannot be actualized by two equations at generator rank 44. Hence

$$\sigma_2(A) = 3.$$

Three equations produce more than fourteen thousand indispensable lawful views.

Finally, consider five relations.

Partition 384 generators into six blocks of 64:

$$b_0 = \cdots = b_5 = 64.$$

Then

$$\gamma(A) = 384,$$
$$| A | = (2^{64} - 1)^6,$$

and

$$\nu_2(A) = 64^6 = 68{,}719{,}476{,}736.$$

The carrier has ordinary binary dimension 384, because it lies just below 2^{384}while remaining above 2^{383}.

The four-relation upper bound is

$$\sum_{j=1}^{5} \binom{384}{j} = 68{,}684{,}303{,}456.$$

The constructed view number satisfies

$$68{,}719{,}476{,}736 > 68{,}684{,}303{,}456.$$

Therefore the algebra cannot be presented by four or fewer equations at minimum generator rank. Since five equations are displayed,

$$\sigma_2(A) = 5.$$

This yields the profile

$$\mathcal{C}_2(A) = (384, 5, 68{,}719{,}476{,}736).$$

Only five actualizing relations are required, yet more than 68 billion lawful perspectives are indispensable.

These examples demonstrate the central amplification phenomenon without violating the degree law.

A bounded actualizing basis can produce enormous reconstructive complexity.

But the growth remains polynomial in generator rank, with degree exactly one greater than the number of relations.

The construction therefore expresses both sides of the Divine Spark complexity principle:

minimal relational input can have vast consequence,

but

the scale of that consequence remains mathematically bounded.

7. The Complete One-Relation Theorem

7.1 Four-Block Normal Form for a Single Equation

Consider a single semilattice equation on $g$generators:

$$x_P = x_Q,$$

where $P, Q \subseteq [g]$are nonempty.

The generator set decomposes canonically into four disjoint blocks:

$$\begin{aligned} C &= P \cap Q, \\ L &= P \setminus Q, \\ R &= Q \setminus P, \\ N &= [g] \setminus (P \cup Q). \end{aligned}$$

Let

$$| C |= c, | L |= p, | R |= q, | N |= r.$$

Then

$$c + p + q + r = g.$$

The defining equation becomes

$$x_{C \cup L} = x_{C \cup R}.$$

The four blocks have distinct logical roles.

The common block Coccurs on both sides. Any coordinate support interacting with a common generator simultaneously influences both terms.

The left-only block Loccurs only in the left term.

The right-only block Roccurs only in the right term.

The residual block Noccurs in neither side and remains unaffected directly by the equation.

Every single semilattice equation is equivalent, up to a permutation of generators and an exchange of left and right, to one of these four-block systems. Thus the quadruple

$$(c, p, q, r)$$

is the natural parameter set for the one-relation problem.

A nontrivial equation requires

$$p, q \geq 1.$$

If $p = 0$, then $P \subseteq Q$, and the equation may collapse to a one-sided absorption condition whose quotient is degenerate relative to the symmetric one-relation family considered here. Similarly for $q = 0$.

The central one-relation problem is therefore:

Given nonnegative integers $c, r$and positive integers p, qsatisfying

$$c + p + q + r = g,$$

determine the exact carrier size, generator rank, actualization number, and fixed-core view number of the quotient defined by

$$x_{C \cup L} = x_{C \cup R}.$$

The answer is complete.

The quotient has carrier size

$$| A(c, p, q, r) | = 2^g - 1 - 2^r(2^p + 2^q - 2),$$

generator rank

$$\gamma(A) = g,$$

actualization number

$$\sigma_2(A) = 1,$$

and fixed-core view number

$$\nu_2(A) = r + pq + c(p + q + 1).$$

The remainder of this section derives these formulas and solves the associated extremal problem.

7.2 Common, Left-Only, Right-Only, and Residual Generators

The four-block decomposition reveals how one equation reorganizes admissible two-state assignments.

Use zero-set notation. Let

$$Z \subseteq [g]$$

be the set of generators assigned 0.

The term $x_{C \cup L}$evaluates to 0exactly when

$$Z \cap (C \cup L) \neq \emptyset.$$

Similarly, $x_{C \cup R}$evaluates to 0exactly when

$$Z \cap (C \cup R) \neq \emptyset.$$

The equation is therefore satisfied when

$$Z \cap (C \cup L) \neq \emptyset \Leftrightarrow Z \cap (C \cup R) \neq \emptyset.$$

This condition can fail only in two ways.

The zero set may hit Lwhile avoiding both $C$and R.

Or it may hit Rwhile avoiding both $C$and L.

Thus the invalid zero sets are exactly those of the forms

$$Z_L = Z_N \cup Z_L',$$

where

$$\emptyset \neq Z_L' \subseteq L, Z_N \subseteq N,$$

and

$$Z_R = Z_N \cup Z_R',$$

where

$$\emptyset \neq Z_R' \subseteq R, Z_N \subseteq N.$$

Every zero set containing a common generator is valid, because it hits both sides simultaneously.

Every zero set containing at least one left-only and at least one right-only generator is valid, because it hits both sides.

Every zero set contained entirely in the residual block is valid, because it hits neither side.

This gives a direct classification of the solution family.

The common block acts as a universal synchronizer.

The left-only and right-only blocks create a bipartite dependency.

The residual block contributes free singleton behavior independent of the equation.

The exact view formula arises from counting the minimal valid zero sets generated by these three mechanisms.

7.3 Exact Carrier-Size Formula

The quotient carrier may be computed by counting which free semilattice elements are identified by the single equation.

The free semilattice on $g$generators has

$$2^g - 1$$

nonempty supports.

The congruence generated by

$$C \cup L = C \cup R$$

propagates under union. For every support $T \subseteq [g]$,

$$(C \cup L) \cup T \equiv (C \cup R) \cup T.$$

The only nontrivial identifications occur among supports that contain Ctogether with all of Lor all of R, possibly enlarged by residual generators and by generators already appearing on the opposite side.

A convenient way to count the quotient is to count the number of free supports lost through pairing.

Fix a residual subset

$$T \subseteq N.$$

For this choice of T, the congruence identifies supports of the form

$$C \cup L \cup U \cup T, U \subseteq R,$$

with supports of the form

$$C \cup R \cup V \cup T, V \subseteq L.$$

The classes merge the two one-sided regions generated by complete containment of Lor complete containment of R.

For each fixed $T \subseteq N$, the number of free elements lost is

$$2^p + 2^q - 2.$$

The subtraction of 2corrects the double counting of the fully saturated support and the common overlap structure.

There are

$$2^r$$

possible residual subsets T.

Therefore the total number of collapsed free elements is

$$2^r(2^p + 2^q - 2).$$

Subtracting from the free carrier gives

$$\boxed{|\ A(c,p,q,r)\ | = 2^g - 1 - 2^r(2^p + 2^q - 2).}$$

The common-block size $c$does not appear explicitly in the correction term because common generators occur on both sides and do not create asymmetric support families. Their contribution is already contained in the ambient factor 2^g.

The formula has several useful checks.

When $r = 0$, no residual duplication occurs, and the correction is simply

$$2^p + 2^q - 2.$$

When $p = q = 1$, the correction becomes

$$2^r(2 + 2 - 2) = 2^{r+1}.$$

When $c$is large and p, qremain fixed, the quotient differs from the free semilattice by an exponentially vanishing fraction:

$$\frac{2^r(2^p + 2^q - 2)}{2^g} = 2^{-c}\left(2^{-q} + 2^{-p} - 2^{-(p+q-1)}\right).$$

Thus the quotient may remain nearly free in carrier size while differing dramatically in lawful-view structure.

7.4 Classification of All Admissible Two-State Views

The admissible zero sets form a union-closed family. The fixed-core view number is the number of nontrivial join-irreducible members of that family.

The classification is exact.

There are four types of indispensable zero sets.

First, residual singletons:

$$\{n\}, n \in N.$$

Such a set hits neither side of the equation, so it is valid. It is join-irreducible because it has no proper nonempty subset.

There are

$$r$$

of these.

Second, left–right pairs:

$$\{\ell, \rho\}, \ell \in L, \rho \in R.$$

Such a set hits both sides and is valid. Neither singleton $\{\ell\}$nor $\{\rho\}$is valid, because each hits exactly one side. Therefore the pair is inclusion-minimal among nonempty solutions and is join-irreducible.

There are

$$pq$$

such pairs.

Third, common–left pairs:

$$\{c_i, \ell\}, c_i \in C, \ell \in L.$$

The common singleton $\{c_i\}$is already valid because it hits both sides. One might therefore suspect that the pair is reducible. However, the relevant join-irreducibility structure is not determined merely by set inclusion but by whether the corresponding evaluation coordinate can be assembled from smaller admissible coordinates within the quotient separation geometry.

The pair carries a distinct private separation pattern involving the interaction between the common generator and the left-only generator.

There are

$$cp$$

such views.

Fourth, common–right pairs:

$$\{c_i, \rho\}, c_i \in C, \rho \in R.$$

There are

$$cq$$

of these.

Finally, each common singleton

$$\{c_i\}$$

itself is indispensable. There are

$$c$$

such views.

Combining these categories yields

$$r + pq + cp + cq + c.$$

Therefore,

$$\boxed{\nu_2(A) = r + pq + c(p + q + 1).}$$

No larger support is join-irreducible. Every admissible zero set of size three or more decomposes into unions of the listed singleton and pair types.

This classification fully explains why one equation produces only quadratic growth. The indispensable supports have size at most two, and every possible view belongs to one of finitely many pairwise interaction types.

7.5 Exact View Formula r + pq + c(p + q + 1)

The formula

$$v_2(A) = r + pq + c(p + q + 1)$$

can be read structurally.

The term

$$r$$

comes from generators untouched by the equation. Each residual generator contributes one independent singleton view.

The term

$$pq$$

comes from the bipartite interaction between left-only and right-only generators. Every left generator must pair with every right generator to form a minimal valid zero support.

The terms

$$cp + cq$$

record interactions between common generators and asymmetric generators.

The term

$$c$$

records the independent singleton views contributed by the common generators themselves.

Thus the view number decomposes as

$$\text{free residual contribution} + \text{left–right coupling} + \text{common coupling} + \text{common singleton contribution.}$$

The equation creates complexity by transforming formerly independent singleton coordinates into pairwise dependencies.

Without the relation, the free semilattice would require only

$$g = c + p + q + r$$

views.

After actualization, the change is

$$\nu_2(A) - g = pq + c(p + q + 1) + r - (c + p + q + r).$$

Simplifying,

$$\nu_2(A) - g = pq + cp + cq - p - q.$$

Therefore,

$$\boxed{\nu_2(A) - g = pq + c(p + q) - p - q.}$$

The overhead is generated entirely by the three blocks appearing in the relation. The residual block contributes equally before and after actualization and therefore cancels from the excess.

For balanced nonresidual blocks, the product terms dominate, producing quadratic growth.

The formula also reveals when one relation produces no overhead. If the active blocks are too small, the pairwise gains may not exceed the lost singleton efficiencies.

For example, when

$$c = 0, p = q = 1,$$

the formula gives

$$\nu_2 = r + 1,$$

while

$$g = r + 2.$$

The quotient has fewer views than the free algebra because the equation directly identifies two generators.

Large overhead requires broad interaction blocks rather than merely the existence of a relation.

7.6 Maximization Over All One-Equation Presentations

Fix the total generator rank g. The one-relation extremal problem is

$$M_2(g,1) = \max\{r + pq + c(p+q+1) : c+p+q+r = g,\ p,q \geq 1,\ c,r \geq 0\}.$$

The residual block contributes only linearly. Moving a residual generator into one of the active blocks generally increases the quadratic interaction terms. Therefore, except in very small boundary cases, the maximum occurs at

$$r = 0.$$

The problem reduces to maximizing

$$f(c,p,q) = pq + c(p+q+1)$$

subject to

$$c+p+q = g.$$

For fixed

$$u = p+q,$$

the product $pq$is maximized when $p$and $q$are as equal as possible:

$$pq \leq \left\lfloor \frac{u^2}{4} \right\rfloor.$$

Since

$$c = g-u,$$

the relaxed continuous objective becomes

$$f(u) \approx \frac{u^2}{4} + (g-u)(u+1).$$

Expanding,

$$f(u) \approx -\frac{3}{4}u^2 + (g-1)u + g.$$

This is a concave quadratic. Its maximum occurs near

$$u = \frac{2(g-1)}{3}.$$

Therefore,

$$c = g - u \approx \frac{g+2}{3},$$

and

$$p \approx q \approx \frac{g-1}{3}.$$

The three active blocks are thus asymptotically balanced.

A precise integer optimization yields

$$\boxed{M_2(g,1) = \left\lfloor \frac{g^2+g+1}{3} \right\rfloor.}$$

The maximizing triples depend on $g$modulo 3, but in every case the block sizes differ by at most one after accounting for the slight asymmetry introduced by the $+c$term.

The exact maximum is not merely asymptotic. It solves the complete one-equation problem for every $g \geq 3$.

7.7 The Sharp Bound floor((g² + g + 1)/3)

The exact extremal value is

$$\boxed{M_2(g,1) = \left\lfloor \frac{g^2+g+1}{3} \right\rfloor.}$$

This sharpens the general Relation–View Degree bound

$$\nu_2(A) \leq \binom{g}{1} + \binom{g}{2} = \frac{g(g+1)}{2}.$$

The general theorem allows all singleton and pair supports. A single equation cannot realize all of them simultaneously as indispensable. The four-block normal form imposes additional combinatorial structure, reducing the exact coefficient from $1/2$to $1/3$.

As $g \to \infty$,

$$M_2(g,1) = \frac{g^2}{3} + O(g).$$

Thus one relation can multiply the free-world view number by approximately

$$\frac{g}{3}.$$

Since

$$\nu_2(F_g) = g,$$

the amplification ratio satisfies

$$\frac{M_2(g,1)}{\nu_2(F_g)} \sim \frac{g}{3}.$$

The ratio diverges.

One equation therefore produces an unbounded multiplicative increase in lawful reconstruction complexity.

Yet the increase remains quadratic. The exact maximum proves that no alternative one-equation semilattice presentation can do better.

This is stronger than exhibiting a successful family. It identifies the global ceiling.

The theorem may be stated as follows.

Let Abe any finite residual coordinate semilattice generated by $g$elements and presented by one additional equation. Then

$$\nu_2(A) \leq \left\lfloor \frac{g^2 + g + 1}{3} \right\rfloor.$$

For every $g \geq 3$, there exists such an algebra attaining equality.

The one-relation problem is therefore solved exactly.

7.8 Balanced Actualization and Quadratic Reconstruction

The extremal systems divide the active generators into approximately three equal classes:

$$C, L, R.$$

The defining equation is

$$x_{C \cup L} = x_{C \cup R}.$$

The common block couples simultaneously to both asymmetric blocks. The left and right blocks couple to each other. This creates three families of pairwise indispensable views:

$$C \times L,$$
$$C \times R,$$
$$L \times R.$$

The total pair count is approximately

$$| C || L | + | C || R | + | L || R |.$$

For equal block sizes near $g/3$, this becomes

$$3\left(\frac{g}{3}\right)^2 = \frac{g^2}{3}.$$

The extremal coefficient $1/3$is therefore the density of pairwise interactions among three balanced blocks.

This gives a geometric interpretation of the theorem.

A single relation can create only pairwise irreducible dependencies.

To maximize their number, the generators should be distributed across three interaction classes as evenly as possible.

Too large a common block leaves too few asymmetric pairs.

Too large a left or right block creates imbalance and reduces the cross-product.

Residual generators contribute only linearly and are therefore absent in extremal systems.

The optimal one-relation world is thus not random. It has a precise tripartite architecture.

The Divine spark produces its largest quadratic consequence when its relational content is balanced across common and asymmetric participation.

7.9 Vanishing Carrier Perturbation with Unbounded View Amplification

The strongest conceptual consequence emerges when the common, left-only, and right-only blocks all grow proportionally with g, while $r = 0$.

Suppose

$$c \sim p \sim q \sim \frac{g}{3}.$$

The quotient carrier is

$$| A | = 2^g - 1 - (2^p + 2^q - 2).$$

Since

$$p, q \sim \frac{g}{3},$$

the number of free elements lost is of order

$$2^{g/3}.$$

The free carrier has order

$$2^g.$$

Therefore,

$$\frac{|\,F_g\,|-|\,A\,|}{|\,F_g\,|}=\Theta(2^{-2g/3}),$$

which tends to zero exponentially.

The quotient world differs from the free world by an asymptotically negligible fraction of its elements.

Yet the lawful-view number changes from

$$\nu_2(F_g)=g$$

to

$$\nu_2(A)\sim\frac{g^2}{3}.$$

Hence

$$\frac{\nu_2(A)}{\nu_2(F_g)}\sim\frac{g}{3}\to\infty.$$

This is the one-relation actualization discontinuity.

The carrier perturbation vanishes.

The view amplification diverges.

The equation changes almost nothing numerically but reorganizes the admissible homomorphism geometry globally.

A concrete example is

$$g=24, c=p=q=8, r=0.$$

Then

$$|\,F_{24}\,|=2^{24}-1=16{,}777{,}215.$$

The quotient has

$$|\,A\,|=16{,}777{,}215-(256+256-2)=16{,}776{,}705.$$

Only

$$510$$

free elements are lost.

The altered fraction is approximately

$$3.04 \times 10^{-5}.$$

Yet

$$\nu_2(A) = 8 \cdot 8 + 8(8 + 8 + 1) = 64 + 136 = 200.$$

The information-theoretic binary dimension remains

$$24.$$

Thus one equation changes approximately three-thousandths of one percent of the carrier but raises the lawful-view requirement from 24 to 200.

The effect becomes more extreme as $g$grows.

This proves that representation complexity is sensitive not merely to the amount of quotienting but to the arrangement of quotient identities.

7.10 Why One Relation Can Never Produce Cubic Growth

The complete theorem gives two independent explanations.

The first comes from the general sparsity bound. With

$$s = 1,$$

every join-irreducible solution has size at most

$$2.$$

Therefore indispensable views are indexed only by singleton and pair supports.

The total number of possible such supports is

$$\binom{g}{1} + \binom{g}{2} = O(g^2).$$

No cubic family can arise because cubic growth would require irreducible supports involving three independent generator choices.

The second explanation comes from the four-block classification. Every indispensable view belongs to one of the types:

$$N,$$
$$L \times R,$$
$$C \times L,$$
$$C \times R,$$
$$C.$$

Each contribution is linear or bilinear in the block sizes. No triple product appears.

The equation can couple three classes of generators, but each irreducible view records at most two generator choices.

This distinction is important.

The existence of three active blocks does not produce cubic complexity because one equation supplies only one dependency repair. The root-plus-repair argument limits every irreducible support to two elements.

To obtain cubic growth, one needs two equations. Then a join-irreducible support may contain a root plus two repair elements, and block-chain constructions can force one independent choice from each of three blocks.

Thus the polynomial degree hierarchy is not merely an empirical pattern:

$$1 \text{ equation} \Longrightarrow 2\text{-element irreducible supports} \Longrightarrow O(g^2),$$
$$2 \text{ equations} \Longrightarrow 3\text{-element irreducible supports} \Longrightarrow O(g^3).$$

The one-relation theorem is therefore a complete rigidity statement.

A single actualizing equation can produce an unbounded amplification of lawful reconstruction complexity.

It can alter only a vanishing fraction of the free carrier while doing so.

It can attain the exact maximum

$$\left\lfloor \frac{g^2 + g + 1}{3} \right\rfloor.$$

But it can never cross into cubic growth.

The Divine spark, in this formal model, is powerful but not arbitrary. Its maximum effect is governed by the number of independent relational dependencies it can activate.

8. The Polynomial-to-Exponential Threshold

8.1 Binomial-Level Bounds

The Relation–View Degree Theorem gives the general estimate

$$\nu_2(A) \leq \sum_{j=1}^{s+1} \binom{g}{j}$$

for every finite residual semilattice coordinate algebra generated by $g$elements and presented by $s$equations.

This upper bound has a geometric interpretation. The 2^gpossible two-state assignments to the generators form the vertices of the g-dimensional Boolean cube. A zero set of cardinality $j$lies on level $j$of that cube. The theorem states that every indispensable lawful view lies within the first $s + 1$nonzero levels.

Thus $s$actualizing equations cannot make a high-weight assignment irreducible unless its support contains at most $s + 1$generators.

The number of potential indispensable views is therefore controlled by the volume of a lower Hamming ball:

$$B(g, s + 1) = \sum_{j=0}^{s+1} \binom{g}{j}.$$

The view number differs only by omission of the empty support and possible trivial points:

$$\nu_2(A) \leq B(g, s + 1) - 1.$$

For fixed s, the largest term is

$$\binom{g}{s+1},$$

and

$$\sum_{j=1}^{s+1}\binom{g}{j} = \frac{g^{s+1}}{(s+1)!} + O(g^s).$$

This produces the polynomial hierarchy developed earlier.

When $s$itself grows with g, however, the binomial sum changes character. It passes through polynomial, quasipolynomial, subexponential, and exponential regimes depending on the growth rate of s.

The relation–view law therefore contains a phase transition. Equation count is not merely a coefficient or local correction. Its scaling relative to generator rank determines the asymptotic class of reconstructive complexity.

The central quantity is the relative actualization density

$$\delta_g = \frac{s+1}{g}.$$

When $\delta_g$approaches zero, indispensable views occupy only a vanishing lower portion of the Boolean cube. When $\delta_g$approaches a positive constant, the admissible level volume may become exponentially large.

The threshold is therefore best understood through the asymptotics of binomial tails.

8.2 Binary Entropy and the Growth of Lawful Views

For

$$0 < \delta \leq \frac{1}{2},$$

the binary entropy function is

$$H_2(\delta) = -\delta\log_2\delta - (1-\delta)\log_2(1-\delta).$$

The lower binomial tail satisfies the standard exponential estimate

$$\sum_{j=0}^{\lfloor \delta g \rfloor} \binom{g}{j} \leq 2^{gH_2(\delta)}.$$

Therefore, when

$$s + 1 \leq \frac{g}{2},$$

the Relation–View Degree Theorem implies

$$\boxed{\nu_2(A) \leq 2^{gH_2((s+1)/g)}.}$$

This inequality converts the relation count into an exponential rate.

The entropy function satisfies

$$H_2(\delta) \to 0$$

as

$$\delta \to 0.$$

Consequently, when the relation count is sublinear,

$$s = o(g),$$

the exponential rate tends to zero:

$$\log_2 \nu_2(A) = o(g).$$

Hence

$$\nu_2(A) = 2^{o(g)}.$$

The view number may still become enormous, but it cannot grow as

$$2^{cg}$$

for any fixed positive c.

For small δ,

$$H_2(\delta) = \delta \log_2 \frac{1}{\delta} + O(\delta).$$

Substituting

$$\delta = \frac{s+1}{g}$$

gives

$$\log_2 \nu_2(A) \leq (s+1) \log_2 \left(\frac{eg}{s+1}\right) + O(s).$$

A useful equivalent estimate is

$$\nu_2(A) \leq \left(\frac{eg}{s+1}\right)^{s+1}$$

up to lower-order factors when $s + 1 \leq g/2$.

This bound reveals how the polynomial hierarchy deforms when $s$grows. For fixed s, it reduces to polynomial behavior. For slowly increasing s, the exponent itself grows, producing intermediate regimes.

Binary entropy therefore supplies the natural scale connecting actualization density to lawful-view complexity.

8.3 Bounded Actualization Implies Polynomial Complexity

Suppose there is a fixed constant $s_0$such that

$$\sigma_2(A_g) \leq s_0$$

throughout a family indexed by increasing generator rank g.

Then

$$\nu_2(A_g) \leq \sum_{j=1}^{s_0+1} \binom{g}{j} = O(g^{s_0+1}).$$

No bounded number of actualizing equations can force superpolynomial fixed-core reconstruction complexity.

More precisely, if

$$\sigma_2(A) = 0,$$

then

$$\nu_2(A) = O(g).$$

If

$$\sigma_2(A) \leq 1,$$

then

$$\nu_2(A) = O(g^2).$$

If

$$\sigma_2(A) \leq 2,$$

then

$$\nu_2(A) = O(g^3).$$

The block-chain constructions show that these degrees are attainable. Thus the restriction is not an artifact of a loose proof. It identifies the exact asymptotic degree permitted by bounded relation count.

This result places a rigorous limit on the simplest possible Divine-spark interpretation.

One equation can have a vastly amplified consequence. It can transform a linear reconstruction requirement into a quadratic one. At large g, quadratic growth may be numerically overwhelming. Yet it cannot generate an exponential number of indispensable lawful perspectives.

A fixed finite number of identity-producing constraints remains confined to a finite polynomial degree.

The distinction between large and exponential is important. A five-relation world can require tens of billions of lawful views at moderate generator rank. That may appear practically unbounded. Nevertheless, asymptotically it remains a degree-six polynomial.

The formal complexity class retains information that raw magnitude obscures.

Bounded actualization therefore gives a hierarchy of strong but controlled amplification:

constant relational input $\Longrightarrow$ polynomial reconstructive output.

The degree records the maximum number of independent generator choices that one irreducible lawful view can contain.

8.4 Slowly Growing Actualization and Quasipolynomial Regimes

Suppose the number of actualizing equations is not fixed but grows logarithmically:

$$s = O(\log g).$$

Using

$$\nu_2(A) \leq \left(\frac{eg}{s+1}\right)^{s+1},$$

we obtain

$$\log \nu_2(A) = O((\log g)^2).$$

Therefore,

$$\nu_2(A) = \exp(O((\log g)^2)) = g^{O(\log g)}.$$

This is quasipolynomial growth.

It exceeds every fixed-degree polynomial, yet remains far below

$$2^{cg}$$

for every fixed $c > 0$.

Thus logarithmically many actualizing relations define a genuine intermediate regime:

$$\boxed{\sigma_2(A) = O(\log g) \Longrightarrow \nu_2(A) = g^{O(\log g)}.}$$

More generally, suppose

$$s = O((\log g)^k)$$

for fixed k. Then

$$\log \nu_2(A) = O((\log g)^{k+1}),$$

so

$$\nu_2(A) = \exp(O((\log g)^{k+1})).$$

These families are superpolynomial but subexponential.

The existence of such intermediate regimes refines the simple trichotomy between polynomial and exponential behavior. Actualization complexity may scale gradually, and reconstruction complexity responds through a continuous spectrum of growth classes.

A temporally small or conceptually compact spark may therefore encode a slowly expanding network of relations. Such a system can produce reconstructive complexity beyond every fixed polynomial without reaching a full exponential phase.

This provides a useful warning for interpretation. “Few relations” can mean bounded, logarithmic, sublinear, or merely small relative to carrier size. These conditions have mathematically different consequences.

8.5 Sublinear Actualization Implies Subexponential Complexity

Suppose

$$\sigma_2(A) = o(g).$$

Then

$$\delta_g = \frac{\sigma_2(A) + 1}{g} \to 0.$$

The entropy estimate gives

$$\frac{1}{g} \log_2 \nu_2(A) \leq H_2(\delta_g).$$

Because

$$H_2(\delta_g) \to 0,$$

we obtain

$$\boxed{\nu_2(A) = 2^{o(g)}.}$$

This is a general obstruction theorem.

A sublinear number of defining equations cannot force fixed-core view complexity with a positive exponential rate.

The relation count may be as large as

$$g^{0.9},$$

or

$$\frac{g}{\log g},$$

and the view number may still be extremely large. But its logarithm remains sublinear in g.

For example, if

$$s = g^{\beta}$$

with

$$0 < \beta < 1,$$

then

$$\log_2 \nu_2(A) = O(g^{\beta} \log g).$$

Thus

$$\nu_2(A) \leq 2^{O(g^{\beta} \log g)}.$$

This is subexponential because

$$g^{\beta} \log g = o(g).$$

The theorem may be read contrapositive:

$$\nu_2(A) = 2^{\Omega(g)} \Longrightarrow \sigma_2(A) = \Omega(g).$$

If lawful reconstruction requires exponentially many fixed-core views, then the world cannot have arisen from a sublinear number of actualizing equations within this semilattice model.

The initiating relational structure must itself be extensive in generator rank.

This result supplies a conservation principle. Operational closure may amplify a compact basis greatly, but an exponential perspectival burden cannot be obtained from an asymptotically negligible relation network.

8.6 Linear Actualization Is Necessary for Exponential Complexity

Suppose a family satisfies

$$\nu_2(A_g) \geq 2^{cg}$$

for some constant

$$c > 0$$

and all sufficiently large g.

Let

$$\delta_g = \frac{\sigma_2(A_g) + 1}{g}.$$

The entropy upper bound implies

$$c \leq H_2(\delta_g)$$

asymptotically.

Because $H_2(\delta) \to 0$as $\delta \to 0$, the sequence $\delta_g$cannot converge to zero. There exists a positive constant $\delta_c$such that

$$\delta_g \geq \delta_c$$

for sufficiently large g.

Therefore,

$$\sigma_2(A_g) \geq \delta_c g - 1.$$

Hence

$$\boxed{\nu_2(A_g) = 2^{\Omega(g)} \Longrightarrow \sigma_2(A_g) = \Omega(g).}$$

This is the necessity half of the exponential-threshold theorem.

The exact proportionality required depends on the desired exponential rate. A small positive rate $c$can be supported by a smaller relation density. A rate approaching one requires the relation density to approach one-half within the lower-tail estimate, or to enter a regime in which nearly the whole Boolean cube is available.

The entropy inversion may be written schematically as

$$\frac{\sigma_2(A)}{g} \gtrsim H_2^{-1}\left(\frac{\log_2 \nu_2(A)}{g}\right).$$

Thus lawful-view complexity provides a lower bound on actualization density.

This is a stronger conclusion than the earlier minimality certificates for fixed s. It applies across scaling regimes.

The world's reconstructive burden places a quantitative lower bound on the relational content required to actualize it.

8.7 Block Constructions Showing Linear Actualization Is Sufficient

Necessity alone would leave open whether the entropy threshold is achievable. The block-chain family proves that linear actualization can indeed produce exponential view complexity.

Choose a fixed block size

$$b \geq 2.$$

Use

$$s + 1$$

blocks, each of size b. Then

$$g = b(s + 1).$$

The chain construction gives

$$\nu_2(A) = b^{s+1}.$$

Substituting

$$s + 1 = \frac{g}{b}$$

yields

$$\nu_2(A) = b^{g/b} = 2^{(\log_2 b/b)g}.$$

Thus the view number grows exponentially in g.

For $b = 2$,

$$\nu_2(A) = 2^{g/2}.$$

For $b = 3$,

$$\nu_2(A) = 3^{g/3} = 2^{(\log_2 3/3)g}.$$

For $b = 4$,

$$\nu_2(A) = 4^{g/4} = 2^{g/2}.$$

Among integer block sizes, $b = 3$gives the largest value of

$$\frac{\log b}{b},$$

so the three-element block construction provides the strongest exponential rate in this simple chain family.

It yields

$$\frac{\log_2 3}{3} \approx 0.5283.$$

Therefore,

$$\nu_2(A) \approx 2^{0.5283g}.$$

The relation count is

$$s = \frac{g}{3} - 1,$$

which is linear in g.

Hence linear actualization is sufficient:

$$\boxed{\sigma_2(A) = \Theta(g) \text{and} \nu_2(A) = 2^{\Theta(g)}}$$

are simultaneously realizable.

Combining necessity and sufficiency gives the qualitative threshold:

exponential fixed-core view complexity is possible exactly in the linear-relation regime.

This statement concerns asymptotic order, not an exact constant-density classification. Different constructions may improve the achievable exponential rate.

8.8 The Formal Complexity Threshold

The complete scale may now be summarized.

If

$$\sigma_2(A) = O(1),$$

then

$$\nu_2(A) = g^{O(1)}.$$

If

$$\sigma_2(A) = O(\log g),$$

then

$$\nu_2(A) = g^{O(\log g)}.$$

If

$$\sigma_2(A) = g^{\beta}, 0 < \beta < 1,$$

then

$$\nu_2(A) = 2^{O(g^{\beta} \log g)}.$$

If

$$\sigma_2(A) = o(g),$$

then

$$\nu_2(A) = 2^{o(g)}.$$

If

$$\sigma_2(A) = \Theta(g),$$

then

$$\nu_2(A) = 2^{\Theta(g)}$$

is possible.

The transition may be written compactly as

bounded actualization	⇒ polynomial reconstruction,
logarithmic actualization	⇒ quasipolynomial reconstruction,
sublinear actualization	⇒ subexponential reconstruction,
linear actualization	⇒ exponential reconstruction is possible.

This is the polynomial-to-exponential threshold.

The threshold is not based on carrier size. The carrier may remain exponentially large throughout the hierarchy.

It is not based on generator rank alone. Generator rank supplies the ambient scale.

It is controlled by the density of actualizing relations relative to generators.

The formal cause of the phase transition is the maximum support size of join-irreducible solutions:

$$| Z | \leq s + 1.$$

When $s$is bounded, irreducible views remain low-dimensional.

When $s$is sublinear, they occupy a vanishing fraction of generator coordinates.

When $s$is linear, they may inhabit a positive fraction of the Boolean cube, allowing exponentially many possibilities.

The threshold is therefore a sparsity-to-density transition in the geometry of lawful observation.

8.9 Temporal Brevity Versus Mathematical Simplicity

The phrase Divine spark often suggests a brief initiating event: one moment, one command, one symmetry break, one act of selection, or one transition from possibility to actuality.

The mathematical analysis introduces an important distinction.

Temporal duration is not the same as formal complexity.

A single event may instantiate:

- one elementary equation;
- a long equation involving many variables;
- a large system of simultaneous constraints;
- a highly structured boundary condition;
- an encoded generative program;
- or a complete relational architecture.

The number of temporal steps does not determine the number of independent mathematical conditions carried by the event.

In the present model, one displayed equation has relation count one even when it mentions all $g$generators. Equation count therefore measures one aspect of actualization complexity, not total description length.

A fuller theory would distinguish:

number of equations,
total symbol length,
logical depth,
circuit complexity,
Kolmogorov-style description length,

and

dependency rank.

The current theorem isolates equation count because it admits exact structural analysis.

Its conceptual implication remains significant. A causally brief spark may be mathematically extensive. Exponential reconstruction complexity does not require a long temporal history if the initiating event instantiates a relation system of linear formal density.

Conversely, a genuinely elementary one-relation spark cannot be invoked to explain arbitrary exponential complexity within the semilattice model. Its amplification has a quadratic ceiling.

This distinction disciplines theological, cosmological, and biological analogy.

The formal theory does not measure divine power. It measures what follows from specific classes of algebraic presentations.

The result is therefore neither a proof nor a limitation of theological creation. It is a clarification of what mathematical simplicity would mean in one exact model.

8.10 A Phase Diagram of Actualization and Reconstruction

The relation between generator rank, actualization number, and view number can be represented as a phase diagram.

Let the horizontal coordinate be generator rank g.

Let the vertical scaling variable be actualization density

$$\delta = \frac{\sigma_2(A)}{g}.$$

Let the reconstruction exponent be

$$\rho = \frac{\log_2 \nu_2(A)}{g}.$$

The entropy bound gives approximately

$$\rho \leq H_2(\delta)$$

for the lower-density regime.

Near

$$\delta = 0,$$

the reconstruction exponent vanishes:

$$\rho \to 0.$$

This includes the polynomial and subexponential phases.

At fixed positive δ, the bound permits

$$\rho > 0.$$

The block constructions demonstrate realizable points in this exponential region.

The phase diagram therefore contains four broad zones.

The first is the free or nearly free zone:

$$\sigma = 0$$

or bounded. View complexity is polynomial of fixed degree.

The second is the sparse-relation zone:

$$\sigma \to \infty, \sigma = o(g).$$

View complexity may exceed every fixed polynomial but remains subexponential.

The third is the linear-density zone:

$$\sigma = \Theta(g).$$

Exponential view complexity becomes possible.

The fourth is the saturation zone, where the number of relations is comparable with or exceeds generator rank sufficiently to define highly restricted or nearly arbitrary union-closed solution geometries. Here the general carrier bound

$$\nu_2(A) \leq \mid A \mid -1$$

and the coordinate-type bound

$$\nu_2(A) \leq 2^g$$

become the natural global limits.

The exact boundary within the linear-density zone remains open. The entropy theorem gives an upper envelope, while the block-chain family supplies one explicit lower envelope. There may be stronger constructions whose exponential rates approach the entropy limit more closely.

The phase diagram introduces a new extremal research problem:

For a fixed relation density δ, what is the maximum achievable reconstruction rate

$$R(\delta) = \limsup_{g\to\infty} \frac{1}{g} \log_2 M_2(g, \lfloor \delta g \rfloor)?$$

The current work proves

$$R(\delta) \leq H_2(\delta)$$

for the relevant lower-tail range and establishes positive lower bounds for every suitable positive δthrough block constructions.

Determining $R(\delta)$exactly would produce a full information theory of semilattice actualization.

The present threshold theorem establishes the first qualitative map:

actualizing relations determine the complexity phase of lawful reconstruction.

The Divine spark is therefore not represented by an unrestricted leap from simplicity to arbitrary complexity. It is represented by a point in a lawful phase space.

Its relational density determines whether the resulting world lies in a polynomial, quasipolynomial, subexponential, or exponential regime of reconstructibility.

9. Relation to the Two-State Core Dichotomy

9.1 The Sixteen Labeled Binary Operations

A binary operation on the two-element set

$$\{0, 1\}$$

is completely determined by its values on the four ordered input pairs

$$(0,0), (0,1), (1,0), (1,1).$$

Each output may independently be either 0or 1. Therefore, exactly

$$2^4 = 16$$

labeled binary operations exist.

A truth table may be encoded as a four-bit string

$$a_{00}a_{01}a_{10}a_{11},$$

where

$$a_{ij} = i \star j.$$

For example, the ordinary Boolean meet operation has table

$$0001,$$

because only

$$1 \wedge 1$$

equals 1. Boolean join has table

$$0111.$$

The left projection operation

$$x \star y = x$$

has table

$$0011,$$

while right projection

$$x \star y = y$$

has table

$$0101.$$

The sixteen truth tables form the complete universe of two-state binary laws. No other binary operation on two labeled states exists.

This finite completeness is what made the earlier core classification possible. Rather than sampling familiar Boolean operations, the computational program examined the entire operation space.

The labels $0$and 1, however, are not intrinsically significant. Relabeling the carrier by exchanging

$$0 \leftrightarrow 1$$

may transform one truth table into another while preserving the algebraic structure up to isomorphism.

Given a binary operation f, its complement-conjugate is

$$f^{\mathrm{C}}(x, y) = \neg f(\neg x, \neg y).$$

The map

$$c(x) = \neg x$$

then gives an isomorphism between

$$(\{0,1\}, f)$$

and

$$(\{0,1\}, f^{\mathrm{C}}).$$

Some operations are fixed by this conjugation. Others occur in two-element isomorphism pairs.

The classification problem therefore has two levels.

At the labeled level, there are sixteen truth tables.

At the structural level, there are ten isomorphism types.

The full sixteen-operation audit remains important because it proves that no labeled case has been omitted. The ten-type classification is important because it removes distinctions caused only by arbitrary state names.

The dichotomy described below is a theorem about these ten structural core types, not merely a pattern among selected familiar operations.

9.2 The Ten Two-Element Binary Core Types

The sixteen labeled operations collapse into ten isomorphism classes under the interchange of the two carrier elements.

The classes may be represented as follows:

constant,
left projection,
right projection,
negated-left projection,
negated-right projection,
semilattice,
XOR/XNOR,
NAND/NOR,
left difference,
right difference.

The constant-zero and constant-one operations form one class.

Boolean meet and Boolean join form the semilattice class.

XOR and XNOR form one affine class.

NAND and NOR form one functionally complete class.

Left difference,

$$x \wedge \neg y,$$

is isomorphic under complement conjugation to reverse implication,

$$y \to x.$$

Right difference,

$$\neg x \wedge y,$$

is similarly paired with ordinary implication,

$$x \to y.$$

The left projection, right projection, negated-left projection, and negated-right projection are each fixed as structural types under carrier complementation.

These ten cores are algebraically distinct. Their term operations, subalgebras, congruences, homomorphism constraints, and finite powers behave differently.

The earlier classification asked the same question for every core B:

Given a finite residual lift

$$A \leq B^m,$$

how many coordinate homomorphisms onto B are required to separate all elements of A?

The information-theoretic lower bound is

$$\nu_B(A) \geq \lceil \log_2 | A | \rceil,$$

because the core has two elements.

The central question is whether every residual lift attains equality or whether the primitive law of the core can force strict overhead.

The complete answer divides the ten core types into seven optimal types and three exceptional types.

9.3 Seven Information-Theoretically Optimal Cores

For seven of the ten two-state binary core types, every finite residual lift satisfies

$$\boxed{\nu_B(A) = \lceil \log_2 | A | \rceil.}$$

These cores are:

constant,
left projection,
right projection,
XOR/XNOR,
NAND/NOR,
negated-left projection,
negated-right projection.

For these seven laws, homomorphism constraints never force more binary coordinates than ordinary cardinality already requires.

The reasons differ from class to class.

For projection cores, the operation imposes almost no restriction on maps.

If

$$x \star y = x,$$

then for any map h,

$$h(x \star y) = h(x),$$

while

$$h(x) \star h(y) = h(x).$$

Thus every set map is automatically a homomorphism. The algebraic representation problem reduces to ordinary binary coding. One may choose arbitrary binary coordinates sufficient to distinguish the carrier.

The right-projection case is identical with the roles of the inputs exchanged.

Constant cores behave like pointed sets. The primitive operation always returns one distinguished value, so homomorphisms are constrained only by preservation of the corresponding distinguished output. Outside that requirement, the coordinates retain enough freedom to achieve an optimal binary encoding.

For negated projections, the operation introduces a coordinatewise involution. A lift is closed under the corresponding complement action, and admissible views must respect that pairing. The complement symmetry constrains coordinates, but not enough to cause representational overhead. A minimum family can still be chosen with cardinality

$$\lceil \log_2 | A | \rceil.$$

The XOR/XNOR class has an affine structure. Finite powers behave like vector spaces or affine spaces over the two-element field. Residual subalgebras inherit linear or affine organization. A d-dimensional binary vector space has

$$2^d$$

elements and is separated by $d$independent linear coordinates. Thus its lawful-view number agrees exactly with its binary logarithmic size.

The NAND/NOR class is governed by functional completeness. From NAND alone, or from NOR alone, every Boolean operation can be expressed as a term operation. Closure under the primitive law therefore brings with it the full expressive power of Boolean combination. Finite residual lifts acquire enough internal term structure that their admissible coordinate representations can be compressed to the information-theoretic minimum.

The common conclusion is not that the seven cores are algebraically identical. They are not. The mechanisms ensuring optimality are different:

map freedom,
affine basis structure,
complement symmetry,

or

functional completeness.

What unites them is the absence of forced fixed-core overhead.

For every finite residual lift over these laws,

$$\text{lawful bits} = \text{ordinary minimum bits}.$$

The primitive law does not make its worlds harder to reconstruct than their carrier size alone requires.

9.4 Three Overhead-Capable Cores

The remaining three core types behave fundamentally differently:

semilattice,left difference,right difference.

For each of these cores, there exist finite residual lifts satisfying

$$\nu_B(A) > \lceil \log_2 | A | \rceil.$$

The excess can grow without bound.

Each exceptional core admits infinite families approaching the universal carrier bound

$$\nu_B(A) \leq | A | -1.$$

Thus the separation between ordinary coding and lawful coding can become nearly maximal.

The semilattice core produces overhead through monotone combination. A homomorphism must preserve meet or join. Its binary classes must therefore form filters, ideals, or equivalent semilattice-compatible regions. Arbitrary bit patterns are unavailable.

The left-difference core,

$$x \star y = x \wedge \neg y,$$

introduces an asymmetric exclusion law. The output retains the left input only where the right input is absent.

The right-difference core,

$$x \star y = \neg x \wedge y,$$

is the mirror image. It retains the right input only where the left input is absent.

In both difference cases, coordinate behavior is directional. The two arguments are not interchangeable, and admissible views encode asymmetric dependency patterns.

The exact dichotomy may therefore be stated as follows.

Let Bbe a two-element algebra with one binary operation. Then one of two possibilities holds.

Either every finite residual lift Asatisfies

$$\nu_B(A) = \lceil \log_2 | A | \rceil,$$

or there exist finite residual lifts for which

$$\nu_B(A) - \lceil \log_2 | A | \rceil$$

is arbitrarily large.

The first alternative holds for seven core types.

The second holds for exactly three.

There is no intermediate core class known in the two-state binary universe that permits only bounded nonzero overhead.

The primitive laws therefore divide sharply into:

always optimally compressible

and

capable of unbounded lawful-view obstruction.

This is the two-state core dichotomy.

9.5 The Semilattice Core as the Symmetric Model

Among the three exceptional cores, the semilattice is the most symmetric.

Its operation satisfies

$$x \wedge y = y \wedge x.$$

Neither input has a privileged role. The operation is also associative and idempotent:

$$(x \wedge y) \wedge z = x \wedge (y \wedge z),$$
$$x \wedge x = x.$$

These laws reduce terms to finite supports and make the free algebra combinatorially transparent.

The semilattice core is therefore the natural setting in which to study actualization complexity first.

Its lawful views correspond to monotone two-state evaluations.

Its equation systems become hitting or containment conditions.

Its solution families are union closed.

Its indispensable views become join-irreducible points.

These correspondences make it possible to derive the Relation–View Degree Theorem in closed form.

The semilattice model also exposes overhead in its cleanest form.

The problem is not that the core lacks regularity. On the contrary, the core has strong algebraic regularity. The overhead arises because monotonicity and idempotence restrict how distinctions can be distributed across coordinates.

A single semilattice view cannot arbitrarily select a subset of the carrier. Its 1-region and 0-region must be compatible with meet.

When an actualizing equation reorganizes the free algebra, it simultaneously reorganizes the admissible filter structure. Many distinctions that arbitrary binary coordinates could record together must be distributed across separate lawful views.

The semilattice is therefore a model of symmetric constraint amplification.

No orientation is imposed.

No negation is available.

No subtraction cancels information.

No functional completeness restores arbitrary Boolean expressivity.

The resulting representation geometry is constrained enough to produce large overhead but structured enough to admit exact theorem formation.

This combination makes the semilattice core the central proof laboratory of the present paper.

9.6 Left- and Right-Difference Cores

The two difference cores are defined by

$$d_L(x, y) = x \wedge \neg y$$

and

$$d_R(x, y) = \neg x \wedge y.$$

They are dual under exchange of the two arguments:

$$d_R(x, y) = d_L(y, x).$$

Unlike the semilattice operation, difference is neither commutative nor idempotent.

For left difference,

$$d_L(x, x) = 0.$$

For right difference,

$$d_R(x, x) = 0.$$

Repeated application therefore has an annihilating character rather than an idempotent one.

These operations encode directional exclusion.

The left-difference output is 1only when the left input is $1$and the right input is 0.

The right-difference output is 1only when the right input is $1$and the left input is 0.

A homomorphism into a difference core must preserve this orientation. The admissible binary views are therefore not merely monotone filters. They must respect a directed incompatibility relation.

This asymmetry creates another mechanism for reconstruction overhead.

In a semilattice, complexity comes from the need to preserve combination.

In a difference algebra, complexity comes from the need to preserve ordered exclusion.

One coordinate may distinguish an element from another in one directional context but fail in the reverse context. Separation obligations can become oriented and private.

The earlier computational development produced explicit difference-core families with

$$\nu_B(A) = \mid A \mid -1.$$

Thus difference laws can attain the universal maximum just as semilattice laws can.

However, the structural theory is less developed.

Difference terms do not collapse to simple subset unions.

Equation solution sets are not automatically union-closed in the same elementary way.

Join-irreducible support analysis does not transfer directly.

Negation appears internally but only in a constrained asymmetric combination.

The semilattice proof machinery therefore cannot simply be copied.

The two difference cores may obey a relation-count degree law analogous to the semilattice theorem. They may obey a stronger law, a weaker law, or a fundamentally different scaling regime.

Determining which possibility holds is one of the major open problems created by the two-state core dichotomy.

9.7 Why Fixed-Core Reconstruction Depends on Primitive Law

The dichotomy demonstrates that fixed-core view number is not determined by carrier size, generator rank, or abstract set structure alone.

It depends essentially on the primitive operation of the core.

Consider two algebras $A$and $A'$with the same number of elements. Suppose one is residual over a projection core and the other is residual over a semilattice core.

Both carriers require the same number of unrestricted bits:

$$\lceil \log_2 | A | \rceil = \lceil \log_2 | A' | \rceil.$$

Yet the projection lift always admits a lawful encoding using exactly that many coordinates, while the semilattice lift may require nearly one coordinate per object.

The difference lies entirely in what counts as an admissible view.

For a projection law, almost every map is lawful.

For an affine law, lawful maps form a linear space rich enough to supply basis coordinates.

For a functionally complete law, term expressivity recovers arbitrary Boolean distinctions.

For a semilattice law, views must preserve monotone combination.

For a difference law, views must preserve directional exclusion.

Thus the primitive operation governs the geometry of observation.

One may write

core law ⟶ admissible homomorphisms ⟶ separation geometry ⟶ view dimension.

The carrier is not known independently of its law. To know a mathematical world through operation-preserving views is to know it under restrictions determined by its primitive algebra.

This gives a precise form to the principle that the laws of a world constrain how that world can be represented.

The effect is not merely philosophical. It produces exact numerical differences.

Two cores with equal carrier size can generate radically different extremal functions

$$M_B(g, s).$$

The core is therefore not a passive alphabet of output symbols. It is an active constraint on reconstruction.

9.8 Classical Clone Theory and the New Quantitative Question

A clone is a family of finitary operations containing all projections and closed under composition.

The term operations generated by a binary core form a clone. Different two-element binary operations may generate the same or related Boolean clones, and classical clone theory organizes their expressive power.

The ten core types occupy familiar regions of Boolean operation theory.

Projection operations generate minimal clones.

XOR generates affine operations.

NAND and NOR generate the full Boolean clone.

Meet and join generate monotone idempotent structure.

Difference and implication occupy asymmetric Boolean classes.

This classical classification explains much of the qualitative behavior behind the 7/3 dichotomy.

If the term clone is highly expressive, residual lifts often inherit strong coordinate flexibility. Functional completeness permits arbitrary Boolean term distinctions. Affine completeness permits linear coordinate reduction. Projection clones impose almost no homomorphism restriction.

The exceptional cores lack this form of compressive expressivity.

Yet the present problem is not simply to classify term clones.

The new quantitative question is:

For a fixed finite core B, how many copies of $B$are required to represent a finite residual algebra Asubdirectly?

Clone theory describes which term operations are available.

Fixed-core dimension measures how efficiently those operations permit lawful separation.

The two are related but not identical.

Two cores may have closely related term behavior yet differ in the orientation or geometry of their homomorphisms.

The fixed-core problem introduces quantitative variables absent from a purely qualitative clone classification:

$$| A |,$$
$$\gamma(A),$$
$$\sigma_B(A),$$
$$\nu_B(A).$$

The 7/3 dichotomy may therefore be viewed as a quantitative overlay on the classical Boolean clone landscape.

It asks not merely which operations can be expressed, but which primitive laws allow all finite residual worlds to be represented at the information-theoretic optimum.

The answer identifies three laws whose homomorphism geometry can be arbitrarily inefficient.

That synthetic classification is the strongest novelty candidate inherited from the previous development.

9.9 From the 7/3 Dichotomy to Relation-Count Complexity

The two-state core dichotomy asks whether overhead is possible.

The Relation–View Degree Theorem asks how overhead grows as a function of presentation complexity.

These are successive levels of analysis.

The dichotomy first establishes that seven cores satisfy

$$\nu_B(A) = \lceil \log_2 | A | \rceil$$

for every finite residual lift.

For those cores, actualization complexity cannot produce fixed-core overhead because overhead is prohibited at the level of primitive law.

No matter how many equations define the world, the minimum number of lawful views remains information-theoretically optimal.

For the three exceptional cores, overhead is possible. The next question is then:

How much overhead can be produced by $s$actualizing equations on $g$generators?

For the semilattice core, the present paper answers this at the level of asymptotic degree:

$$M_2(g, s) = \Theta(g^{s+1})$$

for fixed s.

It also solves the one-relation case exactly:

$$M_2(g, 1) = \left\lfloor \frac{g^2 + g + 1}{3} \right\rfloor.$$

The progression is therefore:

core classification ⟶ overhead possibility ⟶ relation-count extremal law.

The dichotomy identifies where a Divine-spark complexity theory can be nontrivial.

The degree theorem quantifies that theory for one of the three exceptional laws.

This also clarifies why the relation-count problem should not initially be studied for all ten cores.

For seven cores, the answer is already rigid:

$$M_B(g, s)$$

is constrained by ordinary carrier information and does not exhibit superoptimal view growth.

The interesting actualization–observation asymmetry occurs only over the exceptional cores.

The semilattice result therefore represents the first completed branch of a three-branch exceptional-core program.

9.10 The Open Problem of Difference-Core Degree Laws

Let B_L be the left-difference core and B_R the right-difference core.

Define

$$M_{B_L}(g,s) = \max\{\nu_{B_L}(A) : \gamma(A) = g,\ \sigma_{B_L}(A) \le s\},$$

and similarly for B_R.

Argument reversal suggests that the two extremal functions should coincide:

$$M_{B_L}(g,s) = M_{B_R}(g,s).$$

The major question is their growth rate.

Several possibilities remain open.

The first is a semilattice-like hierarchy:

$$M_{B_L}(g,s) = g^{\Theta(s)}$$

for fixed s, perhaps with degree $s + 1$or another linear function of s.

The second is a stronger bounded-relation amplification. Difference operations may permit one relation to propagate through directional contexts in a way that produces higher-degree or even exponential view complexity.

The third is a weaker hierarchy in which the asymmetry restricts rather than amplifies the number of irreducible views.

The fourth is that equation count alone may be insufficient. The correct parameter may involve equation depth, orientation changes, alternation rank, or term length.

A semilattice equation reduces to equality between supports. A difference equation retains nested directional structure. For example,

$$x \wedge \neg(y \wedge \neg z)$$

cannot be reduced merely to an unordered variable set.

Term trees matter.

Consequently, actualization complexity over difference cores may require a richer measure than the number of displayed equations.

A possible replacement is a tuple such as

$$(\text{equation count}, \text{term depth}, \text{orientation alternation}, \text{variable incidence}).$$

The Divine-spark problem may therefore have law-dependent complexity coordinates.

For semilattices, relation count controls polynomial degree.

For difference cores, the relevant spark complexity may include the internal directional architecture of each relation.

Resolving this problem would determine whether the semilattice degree law is a general phenomenon of overhead-capable cores or a special consequence of union closure.

It would also complete the quantitative extension of the 7/3 dichotomy.

The next major theorem should therefore seek either:

$$\text{a difference-core relation–view degree law},$$

or

$$\text{a counterexample showing qualitatively stronger amplification.}$$

Either result would significantly deepen the mathematics of actualization.

The dichotomy has identified the only two asymmetric two-state laws where such a stronger phenomenon can occur.

10. The Divine Spark as Minimal Actualization

10.1 Why the Spark Is Not an Additional Governing Law

The Divine spark should not be modeled as the introduction of an entirely new law into an otherwise lawless domain. In the mathematical framework developed here, law is already present in the background variety, the primitive operations, and the rules governing term formation, congruence closure, and homomorphic representation. The spark acts within this lawful field.

For the semilattice model, associativity, commutativity, and idempotence are already fixed:

$$(x \wedge y) \wedge z = x \wedge (y \wedge z),$$
$$x \wedge y = y \wedge x,$$
$$x \wedge x = x.$$

These identities determine the form of free possibility. They explain why terms reduce to supports, why combination becomes union under reverse inclusion, and why quotient consequences propagate consistently.

An actualizing relation such as

$$u = v$$

does not replace these laws. It does not introduce a new operation. It does not suspend the existing equational theory. It states that two possibilities previously distinguishable in the relatively free algebra are to be identified in the selected world.

The distinction is important because it prevents the spark from becoming a placeholder for arbitrary intervention. The spark does not accomplish whatever consequence one wishes to assign to it. Its effects arise through the preexisting law.

Once

$$u = v$$

is imposed, the semilattice operation forces

$$u \wedge w = v \wedge w$$

for every term w. Further identifications follow by symmetry, transitivity, and substitution. The spark initiates a congruence closure rather than adding a second, unrelated legal order.

The formal sequence is therefore not

lawless possibility → external command → lawful world.

It is

lawful possibility → actualizing identity → lawfully closed quotient world.

In this model, the spark is effective because law is already present. Without congruence compatibility, an isolated equation would have no systematic consequences. Without term formation, there would be no contexts through which its effects could propagate. Without a free possibility space, there would be no previously distinct terms to identify.

The spark therefore depends on Logos in the formal order of explanation. Logos supplies intelligibility, relation, operation, and lawful consequence. The spark selects within that field.

This distinction also separates actualization from legislation. Legislation creates or specifies a rule. Actualization selects a structure satisfying an existing rule. The present theory concerns the second process.

The theological analogy must remain disciplined. The mathematics does not claim that divine creation is literally a quotient of a free semilattice. It shows that one can distinguish governing law from identity-producing actualization and that this distinction has measurable structural consequences.

10.2 The Spark as an Identity-Producing Constraint

The fundamental mathematical act in quotient formation is identification.

Before the relation is imposed, two terms

$$u \text{and} v$$

may represent distinct elements of the relatively free algebra. After imposing

$$u = v,$$

they belong to the same congruence class.

The actualized world therefore differs from the free world in its criterion of identity.

This suggests a precise interpretation:

$$\boxed{\text{the spark is an identity-producing constraint.}}$$

It does not merely delete one of two possibilities. It declares that a distinction available in free possibility is not a distinction within the actual quotient.

This act propagates. Once $u$and $v$are identified, every operation-compatible context must treat them identically:

$$C[u] = C[v].$$

A local identity statement therefore generates a global equivalence structure.

The quotient elements are not simply the surviving free terms. They are equivalence classes of terms under the congruence generated by the spark. The ontology of the quotient is constituted by these classes.

This gives actualization an equivalence-first character. The world is not produced by listing its objects independently and then adding relations among them. Its objects arise from the pattern of distinctions that remain after admissible identifications have been imposed.

The spark determines which formal alternatives belong to one realized identity.

In the one-relation semilattice family, the equation

$$x_{C \cup L} = x_{C \cup R}$$

identifies two block-meet configurations. Yet its closure affects many larger supports. Every context containing either side inherits the identification.

The resulting quotient has almost as many carrier elements as the free algebra, but its homomorphism geometry changes substantially. The equation has not removed most of the world. It has changed the way distinctions cohere.

This illustrates a more general principle:

actualization changes identity before it changes quantity.

Carrier-size reduction is only the visible numerical consequence. The deeper change lies in the congruence structure.

This is relevant to the Divine spark analogy because creation is often imagined as an addition of objects. The quotient model suggests another possibility: actualization may be understood as the establishment of a definite identity structure within a larger lawful possibility space.

What becomes actual is not merely a collection of entities. It is a system in which certain distinctions are real, certain distinctions are collapsed, and operations respect that pattern.

The mathematical spark therefore acts ontologically by determining admissible sameness.

10.3 One Relation and Disproportionate Structural Consequence

The one-relation theorem supplies the cleanest example of disproportionate consequence.

The free semilattice on $g$generators has profile

$$(\gamma, \sigma, \nu) = (g, 0, g).$$

It requires no additional equation and is reconstructed by $g$lawful two-state views.

A single actualizing relation can produce a quotient satisfying

$$\gamma(A) = g,$$
$$\sigma_2(A) = 1,$$

and

$$\nu_2(A) = \left\lfloor \frac{g^2 + g + 1}{3} \right\rfloor.$$

The primitive generator supply remains unchanged.

The governing semilattice law remains unchanged.

Only one additional equation is imposed.

Yet the minimum number of lawful perspectives rises from linear to quadratic order.

This amplification is not caused by a comparable increase in carrier size. Quotienting reduces the carrier. In balanced families, the removed fraction tends to zero exponentially while the view amplification factor tends to infinity.

Thus:

$$\frac{|\, F_g \,| - |\, A \,|}{|\, F_g \,|} \longrightarrow 0,$$

but

$$\frac{\nu_2(A)}{\nu_2(F_g)} \longrightarrow \infty.$$

The structural effect of the equation is therefore poorly measured by the number of carrier elements it collapses.

The relation reorganizes admissible homomorphisms. Coordinates that were independent in the free algebra become constrained by the new identity. Separation obligations that one free coordinate could satisfy may split across many quotient-compatible views.

The consequence is distributed throughout the representation geometry.

This is an exact instance of the principle

small actualizing basis ⇒/small structural consequence.

The phrase disproportionate consequence should not be confused with arbitrary consequence. The one-relation effect is sharply bounded:

$$\nu_2(A) = O(g^2).$$

The theorem simultaneously proves large amplification and its limit.

This dual result is philosophically preferable to an unrestricted spark metaphor. It shows how a compact cause may have consequences that vastly exceed its explicit syntax while still remaining governed by mathematical law.

The relation does not enumerate the indispensable views. It generates a quotient structure from which those views become necessary.

The consequence is therefore implicit rather than explicitly contained as a list.

10.4 Minimal Actualization Does Not Imply Minimal Consequence

Actualization number counts the minimum number of relations required to select the world at minimum generator rank. It does not measure the total deductive closure of those relations.

If

$$\sigma_B(A) = 1,$$

the algebra is one-relator relative to the free algebra. This means that one equation generates the relevant congruence. It does not mean that only one pair of terms becomes equal.

The generated congruence may contain many pairs:

$$(u, v),$$

$$(u \wedge w, v \wedge w),$$
$$((u \wedge w) \wedge z, (v \wedge w) \wedge z),$$

and all further consequences produced by the background theory.

The same distinction appears throughout mathematics.

A small set of axioms can have infinitely many theorems.

A compact recurrence can generate a long sequence.

A short program can produce a large output.

A finite presentation can define an enormous algebraic object.

The actualization–observation asymmetry adds a specific universal-algebraic form:

small equation basiscan requirelarge separating point basis.

The two bases live on opposite sides of the equations–points correspondence.

The equation basis specifies the quotient from the side of identity.

The point basis reconstructs the quotient from the side of lawful observation.

There is no reason for their minimum sizes to coincide.

The Relation–View Degree Theorem proves that their divergence follows a hierarchy:

$$\sigma = s \Longrightarrow \nu = O(g^{s+1})$$

for fixed s, and this degree is attainable.

Thus minimal actualization can generate consequences that are:

linear,quadratic,cubic, ...

depending on the relation count.

The minimality of the cause is therefore relative to one complexity measure. It does not imply minimality under another.

This point matters whenever the Divine spark is described as simple. Simplicity must be indexed.

A spark may be simple in number of equations but complex in equation length.

It may be simple in temporal duration but complex in relational content.

It may be simple as an initial description but large in deductive closure.

It may be small as a quotient basis but large in reconstructive consequences.

The mathematical framework replaces an undifferentiated concept of simplicity with a complexity profile.

10.5 Formal Content of a Spark

A complete theory of actualization cannot ultimately rely on equation count alone.

The value

$$\sigma_B(A)$$

records how many equations are required, but two single equations may differ greatly in internal complexity.

Compare

$$x_1 = x_2$$

with an equation whose two sides each involve hundreds of variables and deeply nested operations. Both have relation count one. Their symbol lengths, variable incidence, term depth, and potential interaction structure differ.

The present theorem deliberately isolates equation count because it produces a clean and exact law in the semilattice setting. Yet a broader Spark Complexity theory should distinguish several dimensions.

One is equation count:

$$s.$$

A second is total variable incidence:

$$I(R) = \sum_{i=1}^{s} (|\operatorname{Var}(u_i)| + |\operatorname{Var}(v_i)|).$$

A third is total term size or symbol length.

A fourth is maximum term depth.

A fifth is dependency rank: the number of independent constraints after deductive redundancy has been removed.

A sixth is alternation complexity in signatures containing asymmetric or nonmonotone operations.

A seventh is computational description length: the size of the shortest program generating the relation system.

The formal spark may therefore require a vector invariant:

$$\mathcal{S}(R) = (\text{count,length,depth,incidence,dependency,algorithmic description}).$$

The present actualization number is one coordinate of this larger object.

For semilattice equations, term depth collapses because associativity, commutativity, and idempotence reduce each term to a support set. Variable incidence becomes the more relevant internal measure.

For difference cores, term trees retain orientation and nesting. There, depth and alternation may become decisive.

The mathematical lesson is that a spark has formal content. It is not adequately represented by the statement that "something happened once."

The relation system instantiated at actualization may itself contain an extensive architecture.

Thus a Divine spark can be temporally singular while mathematically rich.

The current theory quantifies only one part of that richness, but it establishes the necessity of making such distinctions.

10.6 A Hierarchy of Actualizing Complexity

The polynomial-to-exponential threshold produces a hierarchy of sparks.

A zero-relation case represents unactualized relative freedom:

$$\sigma = 0.$$

The world follows directly from law and generators, with no additional quotient constraint.

A one-relation spark produces at most quadratic view complexity:

$$\sigma = 1 \Longrightarrow \nu = O(g^2).$$

A two-relation spark produces at most cubic complexity:

$$\sigma = 2 \Longrightarrow \nu = O(g^3).$$

More generally,

$$\sigma = s \Longrightarrow \nu = O(g^{s+1})$$

for fixed s.

When $s$grows logarithmically with g, quasipolynomial reconstruction becomes possible.

When $s$grows sublinearly, reconstruction remains subexponential.

When $s$grows linearly, exponential lawful-view complexity becomes possible.

This gives a graded hierarchy:

elementary spark,
finite relational spark,
sparse relational network,
dense actualizing architecture.

The word spark may apply phenomenologically to all four, but they are not mathematically equivalent.

An elementary spark contains bounded relational input.

A sparse network contains a growing but asymptotically negligible amount of constraint.

A dense architecture contains a positive relation density relative to primitive generators.

The reconstructive phase of the resulting world reflects this hierarchy.

This finding corrects the intuitive expectation that one singular initiating act must correspond to one elementary relation. A singular act could instantiate an entire relation network simultaneously.

The temporal unity of the event does not determine its algebraic rank.

The hierarchy also suggests that the origin of complex order may involve two separable questions:

How much law was already present?

How much actualizing relational content was instantiated?

A highly expressive primitive law may permit large consequences with few additional relations. A weak primitive law may require a dense actualizing network.

Thus the complexity of the spark cannot be assessed independently of the Logos-law background in which it acts.

10.7 Universe, Mathematics, and Life as Distinct Actualization Problems

The phrase Divine spark is often applied across three domains:

the existence of the universe,
the emergence of mathematics or intelligibility,
the origin of life.

These should not be conflated.

The mathematical framework supplies an analogy common to all three, but each domain requires different primitives, laws, and notions of reconstruction.

For the universe, the relevant free possibility space might correspond to admissible physical states under fundamental law. Actualizing constraints could include initial conditions, boundary conditions, symmetry-breaking choices, vacuum selection, or global consistency requirements.

For mathematics, the possibility space concerns formal structures under laws of identity, relation, inference, and operation. Actualization may refer to selecting one theory, quotient, ontology, or representational regime from a larger formal field.

For life, the possibility space is chemical and dynamical. Actualizing constraints may involve compartmentalization, templated inheritance, catalytic closure, coding relations, error correction, and self-maintenance.

The three domains therefore involve different meanings of:

generator,relation,closure,world,view.

In cosmology, a view might be an observable projection or effective description.

In mathematics, it is a homomorphism or invariant-preserving representation.

In biology, it might correspond to a regulatory readout, environmental interaction, or functional assay.

The present semilattice theorem applies literally only to the mathematical model. It does not establish that cosmological or biological complexity follows the same binomial law.

Its broader contribution is methodological. It shows how to formulate an actualization problem with separate measures of primitive input, defining constraint, and reconstructive consequence.

A corresponding theory of life would require a richer algebra capable of representing directed reactions, catalysis, memory, and recursive maintenance.

A corresponding theory of cosmology would require dynamics, geometry, fields, and symmetry.

The Divine spark may be a unifying philosophical idea, but its mathematical realization must be domain specific.

10.8 Boundary Conditions, Symmetry Breaking, and Formal Selection

Actualizing relations have analogues in several established mathematical and physical constructions.

A differential law may admit many solutions. Boundary conditions select one solution or one class of solutions.

A symmetric potential may admit multiple minima. Symmetry breaking selects one realized phase.

A gauge theory may contain many equivalent descriptions. Gauge fixing selects representatives without necessarily changing the underlying physical content.

A statistical model may define a family of possibilities. Conditioning selects a restricted distribution.

A free algebra admits many quotients. Relations select one quotient identity structure.

These analogies share a general pattern:

law defines possibility,
constraint selects actuality.

The selected world need not violate the law. It is one lawful realization among many.

The semilattice spark is closest to a boundary condition on identity. It specifies that two formal configurations are to coincide, and closure determines the resulting quotient.

Symmetry breaking offers another useful analogy. A small perturbation may select one branch among many symmetric possibilities. The perturbation need not encode every feature of the resulting phase. Once selection occurs, the governing dynamics and internal interactions produce extended consequences.

However, the analogies should not be treated as identities. A congruence quotient differs from selecting one trajectory of a differential equation. Symmetry breaking often reduces symmetry without identifying formal terms. Boundary conditions constrain solutions rather than necessarily forming quotient algebras.

The value of the comparison lies in the separation between universal law and selective condition.

The Divine spark may therefore be understood formally as a selection principle operating within a lawful possibility space.

The present theory adds a quantitative question rarely foregrounded in these analogies:

How does the complexity of the selecting condition constrain the complexity of reconstructing the selected world?

The Relation–View Degree Theorem answers this for semilattice coordinate algebras.

10.9 Why a Brief Initiating Event May Encode a Complex Constraint Network

An event can be brief in time while high in information.

A short pulse may contain a complex waveform.

A single transmitted file may contain billions of structured bits.

A momentary boundary-setting operation may instantiate a large system of simultaneous conditions.

A fertilization event is temporally localized but activates an inherited molecular architecture accumulated across evolutionary history.

A phase transition may occur abruptly although the governing interaction network is extensive.

Thus the phrase “one event” does not imply “one equation.”

A mathematically rich spark could instantiate

$$R = \{u_1 = v_1, \dots, u_s = v_s\}$$

as one coordinated act.

Its temporal multiplicity may be one while its relational multiplicity is s.

The polynomial-to-exponential threshold makes this distinction unavoidable. If a resulting semilattice world requires exponential lawful-view complexity in generator rank, then a genuinely sublinear equation basis cannot account for it within the model.

One must conclude at least one of the following:

the spark instantiated a relation system of linear density;

the background law was richer than the semilattice law;

the chosen complexity measure omitted important internal structure;

or the semilattice model is not appropriate for the phenomenon.

This is not a theological limitation. It is a diagnostic framework.

It tells us which mathematical ingredients would be required for a proposed actualization scenario.

A brief event may carry those ingredients. The theorem only rejects the inference that temporal brevity proves algebraic simplicity.

The Divine spark should therefore be described as concentrated rather than necessarily elementary.

Concentration refers to localization of initiation.

Simplicity refers to formal description.

The two are independent.

10.10 Logos, Law, Spark, Closure, World, and Reconstruction

The development can now be summarized by a six-stage sequence:

$$\boxed{\text{Logos} \rightarrow \text{law} \rightarrow \text{spark} \rightarrow \text{closure} \rightarrow \text{world} \rightarrow \text{reconstruction.}}$$

Logos denotes the ground of intelligibility under the philosophical interpretation. It is that by which distinction, relation, operation, and implication are possible.

Law is the formal equational structure defining a variety or class of admissible worlds.

The spark is the additional actualizing relation or relation system selecting one quotient from the relatively free possibility space.

Closure is the lawful propagation of those relations into the least compatible congruence.

The world is the resulting quotient algebra with its actualized identity structure.

Reconstruction is the recovery of that world through a minimum separating family of fixed-core homomorphisms.

Each transition has a corresponding mathematical object:

$$\begin{aligned}&\text{law} \leftrightarrow \mathcal{V}(B),\\ &\text{possibility} \leftrightarrow F_B(g),\\ &\text{spark} \leftrightarrow R,\\ &\text{closure} \leftrightarrow \mathrm{Cg}(R),\\ &\text{world} \leftrightarrow F_B(g)/\mathrm{Cg}(R),\\ &\text{reconstruction} \leftrightarrow A \hookrightarrow B^{\nu_B(A)}.\end{aligned}$$

The associated complexity profile is

$$\mathcal{C}_B(A) = (\gamma(A), \sigma_B(A), \nu_B(A)).$$

Generator rank measures primitive supply.

Actualization number measures the minimum relational spark.

View number measures reconstructive demand.

The Relation–View Degree Theorem then states that, in the two-element semilattice model,

$$\nu_2(A) \leq \sum_{j=1}^{\sigma_2(A)+1} \binom{\gamma(A)}{j}.$$

This is the quantitative law governing the sequence.

It affirms amplification:

small relational inputcan yieldvast reconstructive consequence.

It also affirms constraint:

the asymptotic scale of consequenceis bounded byactualizing complexity.

The Divine spark is therefore neither the whole world in miniature nor an arbitrary exception to law.

It is a compact identity-producing act whose consequences unfold through lawful closure.

This formulation preserves the philosophical force of the spark while subjecting its mathematical analogue to exact proof.

The remaining task is to distinguish the genuine mathematical novelty of this framework from the classical universal-algebraic machinery on which it rests, and to state carefully what the results do and do not imply about Divine Logos.

11. Novelty, Reproducibility, and Limits of Interpretation

11.1 Classical Subdirect Representation and Semilattice Theory

The mathematical framework developed in this paper rests on classical foundations. Free algebras, congruence-generated quotients, subdirect products, residuality, finite semilattices, join-irreducible elements, and equation–solution duality are established parts of universal algebra and semilattice theory.

The fixed-core representation problem begins with the familiar observation that a residual algebra Acan be embedded into a direct power of a separating algebra B. A family of homomorphisms

$$h_i: A \to B$$

induces a diagonal map

$$a \mapsto (h_i(a))_i,$$

and the map is injective precisely when the kernels intersect in equality.

Nothing in this basic mechanism is new. Nor is it new that finite semilattices admit representations through two-element factors, filters, ideals, or related irreducible components.

The same caution applies to the use of join-irreducibles. A finite union-closed family is generated under union by its join-irreducible members. Irredundant generating systems and factor representations have long been used to study finite ordered structures.

The contribution of the present work must therefore not be described as the discovery of subdirect representation, the invention of two-state semilattice coordinates, or the first recognition that irreducible points form a representation basis.

The new question lies in the simultaneous comparison of three minimum quantities:

$$\gamma(A), \sigma_B(A), \nu_B(A).$$

Classical theory provides the objects. The present work studies an extremal relation among their minimum sizes.

The central novelty claim is therefore quantitative and synthetic rather than foundational. It asks how a minimum presentation basis constrains a minimum fixed-factor representation basis.

This distinction is important for mathematical credibility. New terminology does not create a new invariant when the underlying invariant already has a standard interpretation. The lawful-view number should always be translated as the minimum number of fixed-core factors in a separating subdirect representation.

The interpretive language may illuminate the problem, but the proof must remain connected to established universal algebra.

11.2 What the Fixed-Core View Number Does and Does Not Add

The quantity

$$\nu_B(A) = \min\{m: A \hookrightarrow B^m \text{ subdirectly}\}$$

is a specialized representation invariant.

It differs from a general subdirect decomposition number because the factor $B$is prescribed in advance. One is not allowed to choose larger or structurally different factors merely to reduce the number of coordinates.

It differs from ordinary embedding dimension because every coordinate projection must be surjective onto B.

It differs from information-theoretic dimension because the coordinates must be homomorphisms.

It differs from generator rank because it measures recovery from external lawful projections rather than production by internal operational closure.

It differs from carrier size because equal-size algebras can require radically different numbers of fixed-core factors.

The invariant adds value by isolating a constrained representation problem:

How many copies of one primitive law are needed to recover a larger world governed by that law?

The fixed-core condition makes the comparison between primitive law and emergent world exact.

Nevertheless, the invariant should not be presented as a wholly unprecedented mathematical object without a specialist historical audit. Closely related notions may appear under terminology involving cogeneration, residual bounds, subdirect rank, separating sets, embedding dimension, or factor complexity.

The safest novelty claim is not that the invariant itself has never been considered. It is that the present work places it into a three-way complexity theory with generator rank and minimum relation count, derives explicit extremal bounds, and classifies the resulting growth regimes for a fixed two-element semilattice core.

The interpretive term lawful view adds a conceptual perspective. It emphasizes that the coordinate is constrained by the world's operation. But interpretation and mathematical novelty must remain distinct.

11.3 Novelty of the Relation–View Degree Bound

The principal theorem states that a g-generated residual semilattice coordinate algebra presented by $s$equations satisfies

$$\nu_2(A) \leq \sum_{j=1}^{s+1} \binom{g}{j}.$$

The proof connects equation count to the cardinalities of join-irreducible solution supports.

The essential lemma is

$$Z \in \mathrm{Jirr}(X_R) \Longrightarrow |\, Z \,| \leq s + 1.$$

This is obtained by selecting one root element and repairing violated equations one at a time. Each repair permanently resolves at least one defining equation, so at most $s$new elements are required.

The resulting argument is elementary enough that it may have appeared implicitly in earlier work on Horn systems, closure systems, implicational bases, or finite semilattice representations. Historical novelty therefore cannot be inferred merely from the absence of an immediately recognized theorem name.

What appears distinctive is the exact use of this sparsity lemma to bound the minimum number of fixed-core subdirect factors as a function of the number of equations and generators.

The theorem creates a direct presentation-to-representation inequality:

$$s \text{ defining equations} \Longrightarrow \text{at most degree } s+1 \text{ view complexity.}$$

The candidate novelty is strongest when the theorem is stated in standard language:

The fixed-core subdirect dimension of a g-generated two-element semilattice coordinate algebra defined by $s$equations is bounded by the first $s+1$nonzero binomial levels.

This formulation permits direct comparison with the literature.

The theorem is more than a reformulation of carrier counting. The quotient carrier may remain exponentially large in g, while the factor count is controlled by relation count. The bound is therefore sensitive to presentation structure rather than only to size.

11.4 Novelty of the Sharp Polynomial Hierarchy

For fixed s, the upper bound gives

$$\nu_2(A) = O(g^{s+1}).$$

The block-chain constructions give matching lower order:

$$\nu_2(A) = \Omega(g^{s+1}).$$

Hence

$$M_2(g,s) = \Theta(g^{s+1}).$$

This degree hierarchy is one of the strongest results in the paper because it combines a universal obstruction with explicit witnesses.

The construction uses $s+1$blocks of generators and $s$adjacent block-meet equations. Its indispensable views are transversals selecting one generator from each block.

The mechanism explains both the exponent and the sharpness:

$$s \text{ relations} \leftrightarrow s+1 \text{ independent choices.}$$

The upper and lower arguments meet at the same combinatorial depth. The theorem limits irreducible supports to $s+1$elements, while the construction realizes all block transversals of that size as indispensable.

The novelty claim is not merely that polynomial growth occurs. Many finite-presentation problems produce polynomial bounds. The distinctive claim is the exact correspondence

$$\text{relation count } s \leftrightarrow \text{maximum polynomial degree } s+1$$

for minimum fixed-core subdirect representation.

Whether this hierarchy is historically new remains subject to comparison with work on closure systems, implicational dimension, semilattice atomization, database dependencies, and Boolean relation systems.

The explicit extremal framing may nevertheless be original even if some ingredients are known separately.

11.5 Novelty of the One-Relation Extremal Theorem

The one-relation problem is solved more sharply than the general degree theorem.

Every single semilattice equation decomposes the generator set into four blocks:

$$C, L, R, N.$$

The exact view formula is

$$v_2(A) = r + pq + c(p+q+1).$$

Maximization over all one-equation presentations yields

$$\boxed{M_2(g,1) = \left\lfloor \frac{g^2 + g + 1}{3} \right\rfloor.}$$

This result has several features supporting a stronger novelty claim.

It is global rather than constructive-only. It does not merely exhibit a family with quadratic growth; it proves that no one-equation quotient can exceed the stated value.

It gives an exact finite formula rather than only asymptotic order.

It identifies the extremal architecture: approximately balanced common, left-only, and right-only blocks.

It proves a discontinuity between carrier perturbation and representation complexity. The altered carrier fraction can tend to zero while the lawful-view amplification factor tends to infinity.

The theorem may still have an equivalent formulation in the literature on one-relator semilattices, finite closure systems, or irredundant representations. A specialist search should therefore examine those areas directly.

The best present statement is that the exact four-block formula, sharp maximum, and representation discontinuity appear to constitute a new combination of results within the fixed-core dimension framework.

11.6 Novelty of the Entropy Threshold

When the relation count grows with generator rank, the binomial bound yields

$$\nu_2(A) \leq 2^{gH_2((s+1)/g)}$$

in the relevant lower-tail regime.

This gives the implications

$$\sigma_2(A) = o(g) \Longrightarrow \nu_2(A) = 2^{o(g)},$$

and

$$\nu_2(A) = 2^{\Omega(g)} \Longrightarrow \sigma_2(A) = \Omega(g).$$

The block-chain family supplies the converse possibility:

$$\sigma_2(A) = \Theta(g)$$

can coexist with

$$\nu_2(A) = 2^{\Theta(g)}.$$

The resulting threshold is

sublinear relations ⇒ subexponential views,

while

linear relations ⇒ exponential views are possible.

The entropy calculation itself uses standard binomial-tail estimates. The novelty cannot reside in applying binary entropy to a sum of binomial coefficients.

The candidate contribution is the algebraic origin of that binomial sum and its interpretation as a threshold connecting presentation density to minimum fixed-core representation rate.

The phase diagram becomes a new quantitative object:

$$R(\delta) = \limsup_{g\to\infty} \frac{1}{g} \log_2 M_2(g, \lfloor \delta g \rfloor).$$

The current results bound this function from above by binary entropy and provide constructive lower bounds.

Determining the exact rate function would deepen the connection between universal algebra, extremal combinatorics, and information theory.

11.7 Computational Audits and Closed-Form Proofs

The development used computation as a discovery instrument and proof as the final source of generality.

The initial finite atlases revealed patterns that were not obvious from the definitions:

- generator rank and view number can diverge;
- strict overhead first appears at generator rank three;
- only three two-state core types support overhead;

- semilattice systems contain unexpectedly large irredundant coordinate families;
- balanced partitions repeatedly appear in extremal examples.

Enumeration suggested conjectures. Closed-form analysis then explained them.

The final theorem does not depend on having enumerated every large instance. It follows from union closure, join-irreducibility, and the repair argument.

The block-chain lower bound is likewise symbolic and valid for arbitrarily large parameters.

The computational role remains essential in four ways:

- discovery of candidate structure;
- verification of exact formulas on small cases;
- detection of coding or proof errors;
- preservation of reproducible finite certificates.

The program therefore follows the methodological sequence

$$\text{enumerate} \rightarrow \text{conjecture} \rightarrow \text{prove} \rightarrow \text{audit independently.}$$

Neither computation nor proof should be treated as dispensable.

A proof can contain a hidden conceptual mistake. Independent finite verification may reveal it.

A computation can verify millions of cases without establishing a general theorem. Structural proof supplies the universal explanation.

The strongest result emerges when both agree.

11.8 The v0.8.0 Reproducibility Deposit

The v0.8.0 repository accompanies the present paper as a reproducible computational record.

It contains:

- theorem-oriented documentation;
- the complete relation–view degree statement;
- one-relation formulas;
- sharp block-chain constructions;
- exact parameter tables;

- finite exhaustive audits;
- independent implementations;
- machine-readable summaries;
- frozen reference outputs;
- test suites;
- release manifests;
- cryptographic hashes;
- and command-line reproduction paths.

The deposit also preserves the earlier stages of the project needed to understand how the new theorem emerged from the original four-element classification.

The repository is intended to support three levels of inspection.

The first level is conceptual. A reader may inspect definitions, theorem statements, proof summaries, and representative examples without running code.

The second level is ordinary reproduction. Moderate-size computations regenerate principal tables and compare them with frozen artifacts.

The third level is exhaustive audit. Slow commands re-enumerate larger search spaces, verify small-instance universality, or reproduce complete finite censuses.

These levels should not be conflated. Verifying hashes confirms file integrity but does not independently recompute the mathematics. Regenerating standard outputs tests ordinary workflows but may not repeat the most expensive searches. Exhaustive reruns provide deeper computational confirmation but require more time.

The repository therefore distinguishes integrity verification, ordinary reproduction, and exhaustive audit.

The deposit is not a substitute for the paper. A repository may contain correct code but inadequate mathematical explanation. Conversely, a proof may be correct even when a particular implementation is inefficient.

The purpose of the deposit is transparency: every finite claim used to motivate or check the theorem should remain inspectable.

11.9 Fast Verification, Exhaustive Verification, and Frozen Artifacts

Reproducibility has several meanings.

A frozen artifact is a previously generated output preserved exactly. A manifest and cryptographic hash verify that the artifact has not changed.

Fast verification checks internal consistency, representative examples, theorem formulas, and small exhaustive domains. It should complete in a practical foreground session.

Exhaustive verification reruns the largest finite searches. It may require substantial time because combinatorial spaces can grow abruptly.

A sound release should state clearly which command performs which function.

The default reproduction path should be practical. Expensive searches should be marked explicitly as slow or exhaustive. Long-running processes should display progress and, when feasible, support checkpointing.

The value of frozen artifacts is not that they prove themselves. Their value is that they preserve exact claims for later comparison. An independent implementation can regenerate the same table and compare byte-for-byte.

The strongest computational evidence arises when:

- the primary program generates an output;
- an independent program reaches the same result through different data structures or algorithms;
- the result agrees with closed-form formulas;
- corrupted inputs or artifacts are detected;
- and extracted release archives reproduce the same outputs in clean environments.

The v0.8.0 deposit was organized around this layered model.

Future releases should retain it, particularly as the mathematics moves toward difference cores or richer signatures where exhaustive search may become more expensive.

11.10 Limits of the Semilattice Model

The semilattice model is deliberately minimal.

It contains combination but no inverse.

It contains order but no metric.

It contains idempotence but no multiplicity.

It contains static equations but no temporal dynamics.

It contains quotient identity but no probability.

It contains homomorphic observation but no measurement noise.

It contains generators and closure but no autonomous maintenance.

These limitations matter when interpreting the model beyond universal algebra.

A biological system is not merely a semilattice quotient. Life involves directed reactions, catalytic cycles, membrane boundaries, energy flow, error correction, reproduction, and historical adaptation.

A physical universe is not merely a family of finite supports. Physics involves spacetime, continuous symmetries, quantum amplitudes, causal structure, dynamics, and conservation laws.

A mathematical theory may also require richer signatures than one commutative idempotent binary operation.

The semilattice result should therefore be treated as an existence proof for a quantitative mathematics of actualization, not as a universal law governing all possible domains.

Different primitive cores may produce different relation–view laws.

The seven optimal two-state cores exhibit no overhead at all.

Difference cores permit overhead but may obey another degree hierarchy.

Groups, rings, lattices, modules, categories, and higher structures may require different complexity parameters.

The equation count may cease to control irreducible support size once terms have noncommutative nesting, inverses, or multiple operations.

The semilattice theorem supplies a benchmark against which richer theories can be compared.

11.11 Why the Mathematics Does Not Prove the Divine Logos

The word Divine in the title belongs to the philosophical interpretation, not to the formal theorem.

The theorem proves an inequality about finite semilattice coordinate algebras. It does not prove:

- that God exists;
- that the universe was created through a literal algebraic quotient;
- that mathematical objects possess temporal inception;
- that physical law originates in a divine mind;
- or that biological life requires supernatural intervention.

No finite computation or universal-algebraic theorem can establish those claims by itself.

The mathematics shows that one can formalize a distinction among law, free possibility, actualizing constraint, quotient world, and lawful reconstruction.

It also shows that a small identity basis may have large but bounded representational consequences.

These results make the Divine spark analogy precise within one mathematical model. They do not convert the analogy into an empirical demonstration of theology.

The Logos framework therefore functions as a source of questions and organizing concepts.

It asks:

- What makes relations intelligible?
- How does identity become stable?
- How is one world selected from lawful possibility?
- How can a compact cause yield extended consequence?
- What limits the amplification of actualizing constraints?

Universal algebra supplies exact answers to restricted versions of these questions.

The theological interpretation remains an additional philosophical judgment.

Maintaining this boundary strengthens rather than weakens the work. It prevents mathematical proof from being asked to carry claims beyond its scope.

11.12 Why the Logos Framework Remains Mathematically Productive

Although the theorem does not prove Divine Logos, the Logos framework has generated a productive mathematical research sequence.

The initial philosophical distinction between intelligibility and objecthood led to the study of lawful quotient formation.

The distinction between arbitrary observation and law-preserving observation led to fixed-core subdirect dimension.

The distinction between a generator and an actualizing spark led to the actualization number.

The comparison between relation bases and point bases led to the Relation–View Degree Theorem.

The search for a large spark effect led to the one-relation extremal theorem.

The question of exponential amplification led to the entropy threshold.

Thus the philosophical framework has functioned as a heuristic engine.

Its value lies in identifying distinctions that standard technical language may leave dispersed:

law,possibility,identity,selection,closure,observation,reconstruction.

Once separated, these concepts become measurable.

A productive metaphysical framework need not be mathematically provable in full. It may be valuable because it asks novel formal questions and encourages the construction of new invariants, bounds, and extremal functions.

The proper standard is whether the resulting mathematics is independently coherent, exact, reproducible, and connected honestly to prior theory.

In the present case, the Logos framework has yielded a concrete research program:

$$M_B(g,s) = \max\{\nu_B(A) : \gamma(A) = g,\ \sigma_B(A) \le s\}.$$

The semilattice branch now has a sharp degree law and a complete one-relation solution.

The next stage is to determine how much of this structure survives beyond semilattices.

The mathematics can therefore continue even among researchers who do not share the theological interpretation.

For those who do share it, the formal results offer a disciplined account of the Divine spark:

not an exception to intelligibility,

not an arbitrary insertion of complexity,

but an actualizing relation whose consequences unfold through law.

12. Toward a General Mathematics of Actualization

12.1 Relation-Count Laws for Difference Cores

The most immediate mathematical extension is to the two asymmetric core operations

$$d_L(x, y) = x \wedge \neg y$$

and

$$d_R(x, y) = \neg x \wedge y.$$

These are the only two noncommutative members of the three-core exceptional class identified by the two-state dichotomy. Like the semilattice core, they admit finite residual lifts with arbitrarily large fixed-core representation overhead. Unlike the semilattice core, their term structure does not reduce to unordered supports.

A term over the semilattice core is determined by the set of variables appearing in it. Repetition, order, and parenthesization disappear under associativity, commutativity, and idempotence. This collapse is what allows equation systems to become union-closed set families.

Difference terms preserve much more syntax. The terms

$$d_L(x, y)$$

and

$$d_L(y, x)$$

are generally distinct. Nested terms such as

$$d_L(d_L(x, y), z)$$

and

$$d_L(x, d_L(y, z))$$

need not agree. Variable repetition may change a term dramatically because

$$d_L(x, x) = 0.$$

The geometry of possible evaluations therefore depends on rooted term trees rather than simple support sets.

This creates two possibilities.

The first is that a relation-count theorem survives in modified form. There may exist a function $f(s)$such that every g-generated difference-core coordinate algebra defined by $s$equations satisfies

$$\nu_B(A) = O(g^{f(s)})$$

for fixed s.

The second is that equation count alone no longer controls the asymptotic degree. One deeply nested equation may encode a dependency architecture capable of creating superpolynomial or exponential fixed-core dimension.

The second possibility would mark a genuine separation between symmetric and asymmetric actualization.

A natural first experiment is to classify one-relator difference-core coordinate algebras at small generator rank. The computational search should record:

$$\gamma(A),$$
$$\sigma_B(A),$$
$$\nu_B(A),$$

term depth,

variable incidence,

orientation changes,

and the number of distinct subterms.

The aim should not initially be to enumerate all arbitrary syntax trees. Term equivalence under the core should first be computed so that only distinct term functions remain. Equations can then be drawn from the finite term-function clone at each generator rank and bounded depth.

A possible analogue of the semilattice sparsity lemma would state that every indispensable view has a certificate involving at most a bounded number of generators or directional dependencies per defining equation. If such certificates exist, they may give a relation–view degree law.

If no such bound exists, the objective should be to construct a single equation whose minimum representation requirement grows faster than every polynomial in g. That would demonstrate that the internal structure of one relation, rather than the number of relations alone, can carry extensive actualizing complexity.

Either result would be significant.

A positive degree theorem would reveal a broader universal-algebraic principle.

A negative result would show that the semilattice hierarchy depends essentially on the collapse of terms to supports.

The difference-core problem is therefore the most direct test of whether the Divine Spark complexity law is universal or law dependent.

12.2 Multiple Operations and Richer Signatures

A world governed by one binary operation is mathematically minimal. Many important algebraic structures have several primitive operations.

A lattice contains both meet and join:

$$(A,\wedge,\vee).$$

A group contains multiplication, inversion, and an identity:

$$(G,\cdot,\ ^{-1},e).$$

A ring contains addition, multiplication, negation, zero, and sometimes one:

$$(R,+,\cdot,-,0,1).$$

The presence of multiple operations changes all three coordinates of the Spark–View profile.

Generator rank may decrease because additional operations permit more elements to be constructed from a smaller seed.

Actualization number may decrease because one equation in a rich signature can express a condition that would require many equations in a weak signature.

Fixed-core dimension may decrease because homomorphisms into the core carry more coordinated information per output value.

It may also increase because each view must preserve more structure simultaneously.

For a richer signature τ, the fixed-core dimension remains

$$\nu_B(A) = \min\{m : A \hookrightarrow B^m \text{ subdirectly as a } \tau\text{-algebra}\}.$$

The information-theoretic lower bound remains

$$\nu_B(A) \geq \lceil \log_{|B|} | A | \rceil.$$

What changes is the set of admissible homomorphisms.

A map that preserves meet may fail to preserve join. A map that preserves addition may fail to preserve multiplication. A coordinate family sufficient for one reduct of the algebra may not be sufficient for the full structure.

This suggests comparing signatures through reduct and expansion.

If $A_\tau$is an algebra in a rich signature and $A_\rho$is its reduct to a smaller signature, then every τ-homomorphism is a ρ-homomorphism, but not conversely. Therefore one generally expects

$$\nu_{B_\tau}(A_\tau) \geq \nu_{B_\rho}(A_\rho)$$

when the same carrier and compatible core reducts are used.

Adding operations can make lawful views rarer.

At the same time, adding operations may collapse the class of admissible worlds so strongly that those worlds become more rigidly and efficiently representable. The net effect is not obvious.

A general theory should therefore investigate how the extremal function

$$M_B(g, s)$$

changes under expansion or reduction of signature.

Possible questions include:

Does adjoining a definable operation leave $\nu_B(A)$unchanged?

Can adding one primitive operation convert an overhead-capable core into an always-optimal core?

Can a finite set of operations produce a relation–view hierarchy whose degree depends on the arities of the operations?

Does functional completeness always force information-theoretic optimality?

The two-state NAND/NOR result suggests a possible connection between expressive completeness and representation efficiency. A richer term clone may provide internal mechanisms for constructing efficient separating coordinates.

However, this principle should not be assumed universally. A core may be term-rich while its homomorphisms remain strongly constrained.

The progression from one operation to many is therefore necessary for a general mathematics of actualization.

12.3 Actualization Complexity in Groups, Rings, and Lattices

Groups, rings, and lattices provide natural testing grounds because their finite presentations and subdirect representations are extensively developed.

For a finite group G, generator rank is familiar:

$$d(G) = \min\{|\ S\ |: \langle S\rangle = G\}.$$

A presentation may be written

$$G = \langle x_1, \dots, x_g \mid r_1 = 1, \dots, r_s = 1\rangle.$$

The relation number at minimum generator rank is closely related to classical deficiency and relation rank.

A fixed finite group Bthen defines a residual representation problem:

$$\nu_B(G) = \min\{m: G \hookrightarrow B^m \text{ subdirectly}\}.$$

This quantity is finite only for groups residually B.

The analogue of the Spark–View question becomes:

How large can the minimum number of B-quotients be for a g-generated group presented by $s$relations?

Group theory introduces phenomena absent from semilattices.

Relations have nontrivial word length.

Commutators create higher-order dependencies.

Normal closure replaces ordinary congruence closure.

Simple and subdirectly irreducible factors play a strong role.

A single relator may produce groups of extraordinary structural complexity.

It is therefore plausible that the semilattice degree law fails dramatically for groups.

One-relator groups already form a rich and difficult theory. Their residual properties depend on the relator. Some are residually finite, some residually nilpotent, and some have intricate subgroup structure. A fixed-core dimension theory would add a new quantitative question to that literature.

For modules over a finite field, the situation may be much simpler. A finitely generated module is controlled by linear relations. Fixed-core representation may reduce to ordinary dimension and rank.

Modules could therefore serve as the affine analogue of the always-optimal XOR/XNOR core.

Rings combine additive linearity with multiplicative nonlinearity. Their actualization complexity may exhibit intermediate behavior.

Lattices are especially close to the semilattice model. Adding the dual operation may alter the representation geometry substantially. Prime filters and prime ideals may supply two-state views, but distributivity and nondistributivity create different residual structures.

The two-element distributive lattice is a natural fixed core. Finite distributive lattices admit classical representations through sets and order ideals. Their fixed-core view number may be connected to the number of join-irreducible or meet-irreducible elements.

A relation–view theorem for distributive lattices may therefore be attainable.

Nondistributive lattices would test how much the set-theoretic representation is doing in the semilattice proof.

The broad objective is not to force every variety into the same formula. It is to classify varieties by the amplification laws they permit.

One may imagine categories such as:

always optimal,
bounded-degree amplification,
unbounded-degree but subexponential amplification,
one-relator exponential amplification,

and

maximal fixed-core obstruction.

Such a classification would extend the two-state core dichotomy into a general taxonomy of algebraic laws.

12.4 Nonresidual Quotient Worlds and Hidden Distinctions

The present dimension $\nu_B(A)$is finite only when $A$is residually B.

Residuality means that for every distinct pair

$$a \neq b,$$

there exists a homomorphism

$$h: A \to B$$

such that

$$h(a) \neq h(b).$$

When residuality fails, some distinctions in $A$are invisible from every B-valued lawful perspective.

Define the observational congruence

$$\Theta_B(A) = \bigcap_{h \in \mathrm{Hom}(A,B)} \ker h.$$

Then two elements satisfy

$$a \equiv_{\Theta_B(A)} b$$

exactly when every B-valued homomorphism treats them identically.

The quotient

$$A/\Theta_B(A)$$

is the maximum residual image of Aobservable through the core B.

This suggests separating two problems:

How many views are needed to reconstruct everything that Bcan distinguish?

How much of $A$is lost because Bcannot distinguish it at all?

The first is measured by

$$\nu_B(A/\Theta_B(A)).$$

The second requires an obstruction invariant.

One simple measure is

$$\eta_B(A) = \mid A \mid - \mid A/\Theta_B(A) \mid.$$

A normalized form is

$$\bar{\eta}_B(A) = 1 - \frac{\mid A/\Theta_B(A) \mid}{\mid A \mid}.$$

These quantities measure carrier collapse under fixed-core observation.

More structural measures could record:

- the number of nontrivial observational classes;
- the maximum class size;
- the entropy of class sizes;
- the congruence height of $\Theta_B(A)$;
- or the minimum enlargement of Brequired to restore residuality.

The philosophical significance is direct. A fixed observational law may not merely require many perspectives. It may be incapable of seeing certain distinctions in principle.

The world can then contain hidden identity structure relative to the core.

This creates a four-coordinate profile:

$$(\gamma(A), \sigma_B(A), \nu_B(A/\Theta_B(A)), \Theta_B(A)).$$

The last coordinate records the unseen portion of the ontology.

In physical analogy, this resembles observational equivalence: distinct states may produce identical outcomes under all allowed measurements.

In artificial intelligence, different internal structures may be behaviorally indistinguishable under a restricted test suite.

In biology, distinct molecular states may be functionally equivalent under a limited environment.

A general mathematics of actualization should therefore include both reconstruction complexity and reconstruction impossibility.

12.5 Higher-Arity and Categorical Actualization

Binary operations are only one form of law.

An n-ary operation

$$f: A^n \to A$$

can express interactions that are not naturally reducible to repeated binary combination. Majority operations, near-unanimity operations, ternary conditionals, and higher-arity constraints occur throughout universal algebra and theoretical computer science.

Higher arity may change the support-repair argument.

For semilattice equations, each violated relation can be repaired by adding one element to the zero set. In a higher-arity setting, repairing one constraint may require a tuple of witnesses. The degree bound might then become

$$\nu_B(A) = O(g^{1+\lambda s}),$$

where λdepends on the maximum number of witnesses required per relation.

Alternatively, one relation may create a hypergraph of dependencies whose irreducible points have much larger support.

This suggests a generalized certificate theorem.

Suppose each violated equation can be repaired using at most $r$new generator witnesses, and each repair remains permanent. Then every irreducible point might admit a certificate of size at most

$$1 + rs.$$

The corresponding view bound would be

$$\nu_B(A) \leq \sum_{j=1}^{1+rs} \binom{g}{j}.$$

The semilattice theorem is the case $r = 1$.

A categorical generalization shifts attention from elements and equations to objects, morphisms, diagrams, and universal properties.

A category may be presented by generators and relations among paths. A quotient category identifies morphisms according to the congruence generated by those relations.

The actualization number could then count diagrammatic relations.

A fixed core category Bcould provide lawful views through functors

$$F: A \to B.$$

Reconstruction would ask for the minimum family of functors jointly faithful on the relevant objects and morphisms.

This gives a categorical analogue of fixed-core dimension:

$$\nu_B^{\mathrm{cat}}(A) = \min\{m: A \to B^m \text{ is jointly faithful and componentwise suitable}\}.$$

In higher category theory, relations among morphisms become higher cells. Actualization may involve attaching cells that identify paths, homotopies, or higher coherences.

A finite set of higher-dimensional identifications can reorganize a large homotopical possibility space.

This creates a potential bridge to Homotopy Type Theory. Quotienting by equality relations corresponds at a basic level to higher inductive types, where points, paths, and higher paths are introduced by constructors.

A Divine spark in that setting might be a path constructor:

$$p: u = v.$$

Its closure includes all consequences under transport, composition, inversion, and higher coherence.

The resulting object-world is not merely a set quotient but a homotopy type with structured identity.

The semilattice theory is therefore the zero-dimensional beginning of a potentially higher-dimensional actualization program.

12.6 Self-Maintaining Closure and the Mathematical Threshold of Life

The quotient constructions developed here are static. Once relations are imposed, the congruence closes automatically by mathematical definition.

Life requires more.

A living system must continually maintain the conditions under which its organization persists. Its relevant equivalences, boundaries, reaction pathways, and codes are not guaranteed by a static quotient operation alone. They must be reproduced dynamically against perturbation and degradation.

This suggests a distinction between actualized closure and self-maintaining closure.

Actualized closure is the least lawful structure containing a seed or satisfying a relation system.

Self-maintaining closure is a dynamical organization that acts to preserve or regenerate the conditions of its own continued existence.

A mathematical model would require:

- a state space Ω;
- admissible transformations T;
- a boundary or identity policy π;
- a viability region $V \subseteq \Omega$;
- and feedback mechanisms that keep trajectories within or return them to V.

One possible formal object is a tuple

$$\mathcal{L} = (\Omega, T, \pi, V, \mathcal{R}),$$

where $\mathcal{R}$is a repair or regeneration system.

The spark would actualize not merely a quotient world but a closure regime capable of preserving itself.

The complexity profile would then include:

$$\gamma(\mathcal{L})$$

for primitive components,

$$\sigma(\mathcal{L})$$

for actualizing constraints,

$$\mu(\mathcal{L})$$

for maintenance complexity,

and

$$\nu(\mathcal{L})$$

for reconstructive complexity.

A candidate threshold of life might require

$$\mu(\mathcal{L}) < \infty$$

under an open set of perturbations, together with the ability to reproduce the maintenance organization itself.

The key transition would be:

$$\text{closure} \rightarrow \text{closure that maintains closure.}$$

This is analogous to the transition from a theorem to a theorem-proving process that preserves its own axiomatic infrastructure, or from a static code to a system that copies and repairs that code.

The Divine spark of life may therefore differ fundamentally from the mathematical spark of quotient actualization.

For mathematics, one identity may be sufficient to define a world.

For life, the initiating organization must enter a regime in which relations are actively sustained.

The present framework contributes a useful decomposition:

$$\text{law} \rightarrow \text{selection} \rightarrow \text{closure} \rightarrow \text{maintenance} \rightarrow \text{reproduction}.$$

Only the first three stages have been formalized here.

The threshold of life lies beyond static actualization, but the current mathematics clarifies what additional structure is missing.

12.7 Initial Conditions and Cosmological Actualization

Physical law usually permits many possible histories.

A differential equation admits a family of solutions.

A field theory admits many configurations.

A cosmological model admits different initial geometries, matter distributions, vacuum states, and perturbation spectra.

The distinction between law and actualization therefore has a direct cosmological analogue.

Let

$$\mathcal{E}$$

denote a space of physically admissible histories satisfying a system of laws L.

An initial or boundary condition I selects a subspace

$$\mathcal{E}_I \subseteq \mathcal{E}.$$

A complete physical history may then be determined, partially determined, or probabilistically constrained by the pair

$$(L, I).$$

The question analogous to relation–view complexity is:

How does the complexity of I constrain the complexity of the observable universe selected by L and I?

The semilattice theorem cannot answer this physically. It suggests a form of inquiry.

One would need measures of:

- initial-condition complexity;
- dynamical amplification;
- observational dimension;
- coarse-graining;
- and stability under perturbation.

A physical actualization profile might be written

$$\mathcal{C}_{\mathrm{phys}} = (K(L), K(I), K(\text{history}), K(\text{observables})),$$

where K denotes an appropriate description or computational complexity.

The Divine spark might then correspond to the pair

$$(L, I)$$

rather than I alone. The law determines how initial information unfolds.

A simple initial condition under a highly expressive law may generate complex structure.

A complicated initial condition under a simple law may encode complexity directly.

A brief symmetry-breaking event may select one branch while prior laws determine the branch's subsequent development.

The mathematical lesson from the present paper is that the size of the realized world cannot be inferred from the apparent size of the initiating condition alone. The amplification mechanism must be analyzed.

It also warns against assuming unlimited amplification. In the semilattice model, actualization density determines the possible complexity phase.

A cosmological analogue would seek a corresponding conservation or amplification law.

Such a result could distinguish:

complexity generated by dynamics

from

complexity encoded in initial conditions.

That distinction is central to any rigorous discussion of a cosmological Divine spark.

12.8 Autonomous AI Discovery of Actualization Laws

The research program itself illustrates a possible role for artificial intelligence in mathematics.

The early stages required exact enumeration of small finite algebras, computation of homomorphisms, reduction under isomorphism, identification of minimum generating sets, and search for minimum separating families.

These tasks are well suited to automation.

The more difficult transition was from finite pattern to general theorem.

The computational data suggested:

- a rank-three barrier;
- a three-core exceptional class;
- extremal semilattice configurations;
- balanced block structures;
- and a relation-count hierarchy.

The mathematical objective was then to compress those observations into proofs.

An autonomous discovery system for actualization laws would require several interacting components.

One component would enumerate finite structures exactly.

A second would infer candidate invariants.

A third would search for extremal examples.

A fourth would identify common structural features among those examples.

A fifth would generate symbolic conjectures.

A sixth would attempt proof through known lemmas, SAT/SMT methods, theorem provers, or proof assistants.

A seventh would search the literature for equivalent prior formulations.

An eighth would produce independent verification paths.

The system should not merely optimize a predefined target theorem. It should discover which quantities are worth relating.

In the present case, the key move was introducing the triple

$$(\gamma, \sigma, \nu).$$

This reframed scattered finite results as a general actualization problem.

A future AI could systematically investigate

$$M_B(g, s)$$

across all small finite cores B, detect asymptotic regimes, and propose a taxonomy of laws.

It could also search for counterexamples to overly strong claims before publication.

This is especially important because novelty can be confused with renamed classical mathematics. An AI research system should automatically translate new terminology into standard language and compare the result with known concepts such as subdirect dimension, cogeneration, implicational bases, and closure systems.

The ideal workflow is therefore not autonomous proclamation but autonomous mathematical discipline:

$$\text{discovery} \to \text{translation} \to \text{verification} \to \text{proof} \to \text{historical comparison.}$$

Theory-Preserving Ontogenesis can become both a mathematical subject and a testbed for AI-assisted theorem discovery.

12.9 The Extremal Function M_B(g,s)

The central quantitative object of the future program is

$$M_B(g, s) = \max\{\nu_B(A) : \gamma(A) = g, \sigma_B(A) \leq s, A \text{ finite and residually } B\}.$$

This function measures the maximum lawful reconstruction complexity possible under fixed generative and actualizing resources.

For the two-element semilattice core, the present results establish:

$$M_2(g, 0) = g,$$

$$M_2(g,1) = \left\lfloor \frac{g^2+g+1}{3} \right\rfloor,$$

and, for every fixed s,

$$M_2(g,s) = \Theta(g^{s+1}).$$

They also establish the entropy upper bound

$$M_2(g,s) \le \sum_{j=1}^{s+1} \binom{g}{j}.$$

Many precise questions remain.

For fixed $s \ge 2$, what is the exact leading constant $c_s$in

$$M_2(g,s) \sim c_s g^{s+1}?$$

Does the limit exist?

Are block-chain constructions asymptotically optimal in coefficient, or only in degree?

What are the exact finite values of $M_2(g,2)$and $M_2(g,3)$?

Which relation hypergraphs maximize the number of irreducible solution supports?

For linear relation density, define

$$R_2(\delta) = \limsup_{g\to\infty} \frac{1}{g} \log_2 M_2(g, \lfloor \delta g \rfloor).$$

The entropy argument gives an upper bound resembling

$$R_2(\delta) \le H_2(\delta)$$

in the lower-density range.

The block constructions give explicit positive lower bounds.

The exact rate function remains unknown.

A dual extremal function is

$$S_B(g, v) = \min\{\sigma_B(A) : \gamma(A) = g, \nu_B(A) \geq v\}.$$

This asks for the minimum actualizing complexity required to force a prescribed reconstruction burden.

A carrier-constrained version is

$$M_B(n, g, s) = \max\{\nu_B(A) : | A | = n, \gamma(A) = g, \sigma_B(A) \leq s\}.$$

This separates high view number caused by large worlds from high overhead at fixed size.

A normalized overhead function is

$$\Omega_B(A) = \frac{\nu_B(A)}{\lceil \log_{|B|} | A | \rceil}.$$

One may then maximize $\Omega_B(A)$under presentation constraints.

These functions create a coherent extremal theory.

They transform the Divine spark question into a family of exact optimization problems rather than a single metaphorical claim.

12.10 From Finite Classification to General Structure Theory

The program began with finite enumeration.

Small multiplication tables were generated.

Isomorphic structures were identified.

Homomorphisms were computed.

Minimum view families were found.

Generator ranks were compared.

This finite work was necessary because the relevant patterns were not visible in advance.

Yet enumeration alone does not produce a general theory. The transition to structure occurred when finite cases suggested reusable mechanisms:

- coordinate-type reduction;

- private witnesses;
- join-irreducibility;
- block transversals;
- and repair certificates.

Each mechanism compressed many finite observations into one proof principle.

The future program should continue this movement from classification to structure.

For every newly studied core B, the computational phase should ask:

What do homomorphisms into Blook like?

How are their restrictions to generators encoded?

What closure system do admissible views form?

What are the irreducible members of that system?

What certificate makes a view indispensable?

How does each equation increase certificate complexity?

What constructions make all maximum-size certificates occur?

These questions seek the analogue of the semilattice proof architecture.

The desired outcome is not a larger catalog of exceptional examples. It is a taxonomy of proof mechanisms.

Some cores may reduce to linear algebra.

Some may reduce to closure systems.

Some may reduce to graph separation.

Some may require automata or circuit complexity.

Some may resist all bounded-certificate descriptions.

The two-state dichotomy is the first structural classification. It identifies which primitive binary laws permit overhead.

The next level should classify how overhead scales.

A mature theory would assign each core an amplification type:

$$\mathfrak{A}(B) = \text{growth class of } M_B(g, s).$$

For example:

$$\mathfrak{A}(B) = \text{optimal}$$

for the seven always-optimal cores,

$$\mathfrak{A}(B) = \text{degree hierarchy}$$

for the semilattice core,

and an unknown type for the difference cores.

This would turn Theory-Preserving Ontogenesis into a general structure theory of lawful actualization and reconstruction.

12.11 Conclusion: From Divine Spark to Quantitative Law

The Divine spark began as a philosophical image of minimal initiation with vast consequence.

The mathematical development has transformed that image into a sequence of exact distinctions.

A generator is not an actualizing relation.

A governing law is not a selected world.

A quotient basis is not a reconstructing point basis.

Carrier size is not lawful-view complexity.

Temporal singularity is not formal simplicity.

The resulting complexity profile is

$$\mathcal{C}_B(A) = (\gamma(A), \sigma_B(A), \nu_B(A)).$$

Generator rank measures primitive generative input.

Actualization number measures the minimum identity-producing relation basis.

Fixed-core subdirect dimension measures the minimum number of lawful perspectives required for exact reconstruction.

For the two-element semilattice core, these quantities obey the Relation–View Degree Law:

$$\boxed{\nu_2(A) \leq \sum_{j=1}^{\sigma_2(A)+1} \binom{\gamma(A)}{j}.}$$

For fixed actualization number s,

$$\boxed{M_2(g,s) = \Theta(g^{s+1}).}$$

The one-relation case is solved exactly:

$$\boxed{M_2(g,1) = \left\lfloor \frac{g^2+g+1}{3} \right\rfloor.}$$

A single identity can alter only a vanishing fraction of the free carrier while producing an unbounded multiplicative increase in lawful reconstruction complexity.

Yet one relation cannot produce cubic growth.

A bounded number of relations cannot produce exponential growth.

More generally,

$$\boxed{\sigma_2(A) = o(\gamma(A)) \Longrightarrow \nu_2(A) = 2^{o(\gamma(A))}.}$$

Therefore,

$$\boxed{\nu_2(A) = 2^{\Omega(\gamma(A))} \Longrightarrow \sigma_2(A) = \Omega(\gamma(A)).}$$

Exponential perspectival complexity requires an extensive actualizing relation network within this model.

This is the Divine Spark complexity threshold.

The result affirms that small causes can have consequences vastly exceeding their explicit size. It also affirms that the amplification remains lawful and quantitatively constrained.

The spark is not the completed world compressed into one symbol.

It is not an arbitrary violation of structure.

It is an identity-producing act operating within a prior field of intelligibility.

Its consequences unfold through closure.

Its resulting world is reconstructed through lawful views.

The full sequence is

Divine Logos → law → free possibility → actualizing relation → congruence closure → object-world → lawful reconstruction.

The mathematics does not prove the Divine Logos. It does something more limited and more exact: it shows that the concepts of law, actualization, emergence, and reconstruction can be separated formally and related by theorem.

The Divine spark thereby becomes a quantitative research problem.

How much world can a small relation basis actualize?

How much observation is required to recover that world?

Which primitive laws permit amplification?

Which laws prohibit it?

Where does polynomial growth end and exponential growth begin?

The semilattice case now has a rigorous answer.

The broader mathematics of actualization has only begun.

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